Lecture 2: Free Choice and Disjunction

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Team Semantics: Linguistic and Philosophical Applications
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Readings

- Aloni, M. (2022). "Logic and conversation: The case of free choice." *Semantics and Pragmatics*, 15, 1–60. https://doi.org/10.3765/sp.15.5
- Degano, M., Marty, P., Ramotowska, S., et al. (2025). "The ups and downs of ignorance." *Natural Language Semantics*, 33, 1–41. https://doi.org/10.1007/s11050-024-09226-3
- Aloni, M., Anttila, A., & Yang, F. (2024). "State-based modal logics for free choice." *Notre Dame Journal of Formal Logic*, 65(4), 367–413. https://doi.org/10.1215/00294527-2024-0027

Outline

Why disjunction?

- Classical disjunction $\alpha \vee \beta$ is true iff at least one disjunct is true.
- Natural-language disjunction does more: it presents alternatives against an information state.
- Two related inferences we will focus:
  - ignorance inferences
  - free choice inferences

Ignorance

The dog is in the attic or in the garden.

- The sentence typically suggests that the speaker does not know which location is actual.
- But classical truth conditions do not entail ignorance.
- Even if the speaker *knows* that the dog is in the attic, the disjunction is still true.
- Full ignorance has two separable components:

Possibility

$$\Diamond_e \alpha \wedge \Diamond_e \beta$$

Each disjunct remains compatible with the relevant epistemic state.

Uncertainty

$$\neg \Box_e \alpha \wedge \neg \Box_e \beta$$

Neither disjunct is settled by the relevant epistemic state.

Free Choice

You may go to the beach or to the cinema.

- The hearer normally infers that both options are permitted.
- But in standard modal logic:

$$\Diamond(\alpha \vee \beta) \nvDash \Diamond\alpha \wedge \Diamond\beta$$

- A world may access only an α-world and no β-world.

The Paradox of Free Choice

- Suppose we simply add the free-choice principle:

$$\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$$

- Classical reasoning then gives Ross-style overgeneration:

1.	$\Diamond\alpha$	assumption
2.	$\Diamond(\alpha \vee \beta)$	from 1 by addition and monotonicity
3.	$\Diamond\beta$	by free choice

- Natural language resists the corresponding inference.

  You may post this letter.
  Therefore, you may post this letter or burn it.

Free Choice beyond the Basic Case

- Narrow scope free choice:

$$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$$

- Wide scope free choice:

$$\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$$

- Dual prohibition:

$$\neg \Diamond (\alpha \vee \beta) \rightsquigarrow \neg \Diamond \alpha \wedge \neg \Diamond \beta$$

- Universal free choice:

$$\forall x \, \Diamond (\alpha(x) \vee \beta(x)) \rightsquigarrow \forall x(\Diamond\alpha(x) \wedge \Diamond\beta(x))$$

Outline

Teams Again

- Let $N \subseteq \text{Prop}$ be finite.
- A valuation is a map $v : N \to \{0, 1\}$.
- A team is a set $T \subseteq 2^N$.

T	p	q
v_1	1	1
v_2	1	0
v_3	0	1

Split Disjunction: Lax Clause

$$T \vDash \phi \vee \psi \quad \text{iff} \quad \exists S, U \subseteq T \text{ such that } S \cup U = T,\ S \vDash \phi,\ U \vDash \psi$$

- The team may be decomposed into a ϕ-supporting part and a ψ-supporting part.
- The decomposition need not be unique.
- In the lax version, overlap is allowed.
- Empty witnesses are allowed.

Strict Split Disjunction

$$T \vDash \phi \vee_{\mathrm{str}} \psi \quad \text{iff} \quad \exists S, U \subseteq T \text{ such that } S \cap U = \varnothing,\ S \cup U = T,\ S \vDash \phi,\ U \vDash \psi$$

- Strict split requires a partition of the team.
- Lax split only requires a cover.
- For downward closed logics, the two versions agree: overlap can be removed.

Split Disjunction: Example

T	p	q
v_1	1	1
v_2	1	0
v_3	0	1

- Let $S = \{v_1, v_2\}$ and $U = \{v_1, v_3\}$.
- Then $S \vDash p$, $U \vDash q$, and $S \cup U = T$.
- Thus $T \vDash p \vee q$.

Inquisitive Disjunction

- Boolean/inquisitive/intuitionistic team disjunction, often written $\phi \mathbin{\vee\!\!\!\vee} \psi$, is:

$$T \vDash \phi \mathbin{\vee\!\!\!\vee} \psi \quad \text{iff} \quad T \vDash \phi \text{ or } T \vDash \psi$$

- Split disjunction is weaker (assuming empty team property):

$$T \vDash \phi \mathbin{\vee\!\!\!\vee} \psi \quad \Rightarrow \quad T \vDash \phi \vee \psi$$

- The converse fails.

T	p	q
v_1	1	0
v_2	0	1

$$T \vDash p \vee q \qquad T \nvDash p \mathbin{\vee\!\!\!\vee} q$$

Empty Witness

- $T \vDash \phi \vee \psi$ allows $T = S \cup \varnothing$
- If $T \vDash \phi$, then $T \vDash \phi \vee \psi$ whenever $\varnothing \vDash \psi$.
- Many team logics have the empty team property:

$$\varnothing \vDash \chi \quad \text{for all formulas } \chi$$

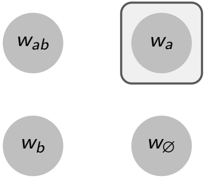
- Thus split disjunction validates a form of addition: $\phi \vDash \phi \vee \psi$

Zero-Models for Disjunction

- A state may verify $a \vee b$ by assigning no worlds to one disjunct.
- Example: $s = \{w_a\}$

  $s = \{w_a\} \cup \varnothing \quad \{w_a\} \vDash a \quad \varnothing \vDash b$

- We call this a zero model.



Non-Empty Split Disjunction

A natural strengthening is:

$$T \vDash \phi \vee^{\mathsf{ne}} \psi \quad \text{iff} \quad \exists S, U \subseteq T \begin{cases} S \cup U = T \\ S \neq \varnothing, \quad U \neq \varnothing, \\ S \vDash \phi, \quad U \vDash \psi \end{cases}$$

- Both disjuncts must have witnesses.
- This removes disjunctions verified only by empty configurations.

Outline

From Propositional Teams to Modal States

- A Kripke model is $M = \langle W, R, V \rangle$.
- A modal state (team) is a set $s \subseteq W$.
- Classical modal logic evaluates formulas at a world: $M, w \vDash \phi$
- State-based modal logic evaluates formulas at a state: $M, s \vDash \phi$

Why Bilateral?

- BSML is interpreted using two relations:

$$M, s \vDash \phi \quad \text{“}s \text{ supports } \phi\text{”}$$

$$M, s \dashv \phi \quad \text{“}s \text{ rejects } \phi\text{”}$$

- Negation swaps support and rejection:

$$M, s \vDash \neg\phi \iff M, s \dashv \phi$$

$$M, s \dashv \neg\phi \iff M, s \vDash \phi$$

Language of BSML

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \Diamond\phi \mid \mathrm{ne} \qquad (p \in \mathrm{Prop})$$

- $\vee$ is split disjunction.
- ne is the non-emptiness atom.
- $\Diamond$ is a relational possibility modality.
- $\Box\phi$ may be introduced as $\neg\Diamond\neg\phi$.

The ne-free fragment is the standard modal language. This fragment is flat and its semantics coincides with the standard semantics on singletons:

$$M, s \vDash \phi \iff M, \{w\} \vDash \phi \text{ for all } w \in s \iff M, w \vDash_{\mathrm{cl}} \phi \text{ for all } w \in s.$$

Bilateral Atomic Clauses

Given $M = \langle W, R, V \rangle$ and $s \subseteq W$:

$$M, s \vDash p \quad \text{iff} \quad \forall w \in s : V(w, p) = 1$$
$$M, s \dashv p \quad \text{iff} \quad \forall w \in s : V(w, p) = 0$$

- Support of p: every live world makes p true.
- Rejection of p: every live world makes p false.

Negation

$$M, s \vDash \neg\phi \quad \text{iff} \quad M, s \dashv \phi$$
$$M, s \dashv \neg\phi \quad \text{iff} \quad M, s \vDash \phi$$

- Negation is defined by the support/rejection duality.
- Double negation elimination is valid $\neg\neg\phi \equiv \phi$.
- But, as we will see, replacement of equivalents under negation does not hold in general: $\phi \equiv \psi$ need not imply $\neg\phi \equiv \neg\psi$.

Disjunction and Conjunction in BSML

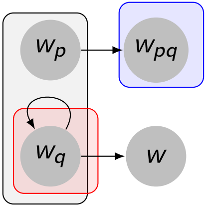
$$
\begin{array}{lll}
M, s \vDash \phi \vee \psi & \text{iff} & \exists t, t' \subseteq s : t \cup t' = s, M, t \vDash \phi, M, t' \vDash \psi \\
M, s \dashv \phi \vee \psi & \text{iff} & M, s \dashv \phi \text{ and } M, s \dashv \psi \\
M, s \vDash \phi \wedge \psi & \text{iff} & M, s \vDash \phi \text{ and } M, s \vDash \psi \\
M, s \dashv \phi \wedge \psi & \text{iff} & \exists t, t' \subseteq s : t \cup t' = s, M, t \dashv \phi, M, t' \dashv \psi
\end{array}
$$

Modality in BSML

For $R[w] = \{v \in W \mid wRv\}$:

$$M, s \vDash \Diamond\phi \quad \text{iff} \quad \forall w \in s\ \exists t \subseteq R[w]\ (t \neq \varnothing \text{ and } M, t \vDash \phi)$$

$$M, s \dashv \Diamond\phi \quad \text{iff} \quad \forall w \in s:\ M, R[w] \dashv \phi$$



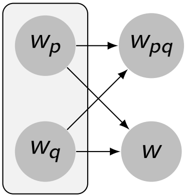
$$s \vDash \Diamond q \text{ since } \begin{array}{l} \{w_q\} \subseteq R[w_q] \\ \{w_q\} \vDash q \end{array} \quad \text{and} \quad \begin{array}{l} \{w_{pq}\} \subseteq R[w_p] \\ \{w_{pq}\} \vDash q \end{array}$$

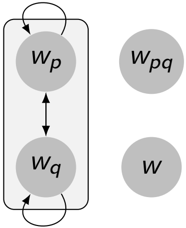
- Possibility is checked pointwise in the current state.
- But the witness for each world is a non-empty substate of its accessible worlds.

Accessibility Constraints

Two team-sensitive constraints on R:

- R is *indisputable* in (M, s) iff $\forall w, v \in s: \ R[w] = R[v]$
- All worlds in the state agree on the relevant modal background.
- R is *state-based* in (M, s) iff $\forall w \in s: \ R[w] = s$
- The accessible worlds are exactly the worlds compatible with the current state.





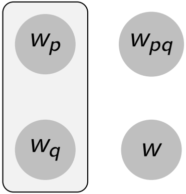
Epistemic vs. Deontic Modals

- If s represents the information state of a speaker indisputability of s means that the speaker is fully informed about the accessibility relation: there is no uncertainty as to what R-accessible.
- State-basedness adds the constraint that what is accessible is the information state s itself.
- Epistemic modality is naturally state-based: $R[w] = s$
- If the epistemic state is s, every live world sees the same epistemic possibilities, and those are precisely the possibilities in s.
- Deontic modality need not be state-based.
- But in performative permissions, the relevant deontic background may be indisputable: the speaker is fully informed about what is permissible.

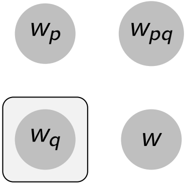
The Non-Emptiness Atom

$$M, s \vDash \text{ne} \quad \text{iff} \quad s \neq \varnothing$$
$$M, s \dashv \text{ne} \quad \text{iff} \quad s = \varnothing$$

- ne is true exactly when the current witness state is non-empty.
- It lets the object language rule out empty witnesses.
- It is the formal handle on neglect-zero effects.



$s \vDash (p \wedge \text{ne}) \vee (q \wedge \text{ne})$



$s \nvDash (p \wedge \text{ne}) \vee (q \wedge \text{ne})$

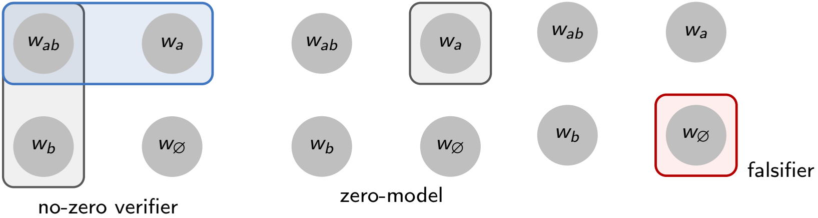
Väänänen & Yang 2014

Outline

The Neglect-Zero Hypothesis

- When interpreting a sentence, speakers construct models or witnesses for its verification.
- Some verification patterns work only because an empty configuration does the work.
- These are zero-models.
- The hypothesis: ordinary interpretation tends to neglect such zero-models.

Verifier, Zero-Model, Falsifier



- A no-zero verifier for $a \vee b$ has a non-empty a-part and a non-empty b-part.
- A zero-model verifies $a \vee b$ only because one witness is empty.
- A falsifier supports neither disjunct.

Neglect-Zero Enrichment

For ne-free formulas, define $[\alpha]^+$ recursively:

$$
\begin{array}{rcl}
[p]^+ & = & p \wedge \mathrm{ne} \\
[\neg\alpha]^+ & = & \neg[\alpha]^+ \wedge \mathrm{ne} \\
[\alpha \vee \beta]^+ & = & ([\alpha]^+ \vee [\beta]^+) \wedge \mathrm{ne} \\
[\alpha \wedge \beta]^+ & = & ([\alpha]^+ \wedge [\beta]^+) \wedge \mathrm{ne} \\
[\Diamond\alpha]^+ & = & \Diamond[\alpha]^+ \wedge \mathrm{ne}
\end{array}
$$

Enriched Disjunction

$$[\alpha \vee \beta]^+ = ([\alpha]^+ \vee [\beta]^+) \wedge \mathsf{ne}$$

- A state supports $[\alpha \vee \beta]^+$ only if it can be covered by two non-empty substates:

$$s = t \cup t' \quad t \vDash [\alpha]^+ \quad t' \vDash [\beta]^+$$

- Empty disjunct-witnesses are excluded.
- Therefore both disjuncts must be live possibilities.

$$[\alpha \vee \beta]^+ \vDash \Diamond_e \alpha \wedge \Diamond_e \beta \quad \text{when } R \text{ is state-based.}$$

But Not Full Ignorance

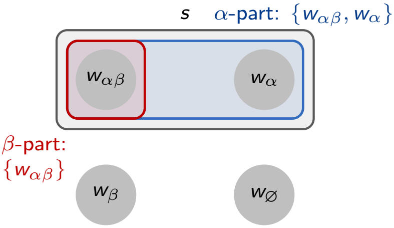
- Enriched disjunction gives possibility:

$$[\alpha \vee \beta]^+ \vDash \Diamond_e \alpha \wedge \Diamond_e \beta$$

- It does not by itself give uncertainty:

$$[\alpha \vee \beta]^+ \nvDash \neg \Box_e \alpha \wedge \neg \Box_e \beta$$

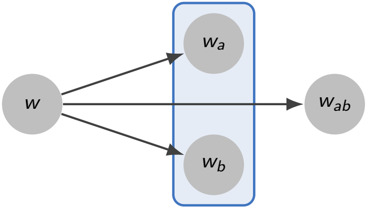
- Neglect-zero derives *weak ignorance*: both disjuncts are possible, although one may already be settled.



$\Diamond_e \alpha$ and $\Diamond_e \beta$, but also $\Box_e \alpha$

Narrow Scope Free Choice

$$[\Diamond(\alpha \vee \beta)]^+ \vDash \Diamond\alpha \wedge \Diamond\beta$$



non-empty split witness for $a \vee b$

- To support $[\Diamond(\alpha \vee \beta)]^+$, each current world must access a non-empty state supporting $[\alpha \vee \beta]^+$.
- That witness itself contains a non-empty α-part and a non-empty β-part.
- Therefore each current world has an α-possibility and a β-possibility.

Why Ross's Paradox Is Avoided

- $\Diamond\alpha \vDash \Diamond(\alpha \vee \beta)$ in the unenriched fragment.
- But free choice is derived for the enriched formula: $[\Diamond(\alpha \vee \beta)]^+$.
- From $\Diamond\alpha$ we do not get $[\Diamond(\alpha \vee \beta)]^+$
- The latter requires a modal witness in which both disjuncts have non-empty support.

Dual Prohibition

$$[\neg \Diamond (\alpha \vee \beta)]^+ \vDash \neg \Diamond \alpha \wedge \neg \Diamond \beta$$

- Under single negation, enrichment with ne is vacuous in the relevant way.
- Rejection of a disjunction is rejection of both disjuncts.
- Therefore no permission for the disjunction yields no permission for either option.

Double Negation Free Choice

$$[\neg\neg \Diamond (\alpha \vee \beta)]^+ \vDash \Diamond\alpha \wedge \Diamond\beta$$

- BSML validates double negation elimination at the level of formulas.
- Single negation suspends the effect. Double negation reintroduces it.

Wide Scope Free Choice

$$[\Diamond\alpha \vee \Diamond\beta]^{+} \vDash \Diamond\alpha \wedge \Diamond\beta \quad \text{if } R \text{ is indisputable.}$$

- Enriched wide-scope disjunction gives a non-empty $\Diamond\alpha$-supporting substate and a non-empty $\Diamond\beta$-supporting substate.
- If all worlds in the state have the same modal background, these witnesses can be transferred across the whole state.
- Hence the whole state supports both possibilities.

Failure of Wide Scope FC without Indisputability

- If worlds in s disagree about the modal background, one part of the state may support $\Diamond\alpha$ and another part may support $\Diamond\beta$.
- The enriched disjunction then holds, but the whole state need not support either modality.

$$[\Diamond\alpha \vee \Diamond\beta]^+ \not\models \Diamond\alpha \quad \text{when } R \text{ is not indisputable.}$$

- This predicts contexts where overt free-choice cancellations are possible.

Epistemic and Deontic Predictions

- Epistemic modals: *R* is state-based, hence indisputable.

$$\text{Might } \alpha \text{ or might } \beta \rightsquigarrow \text{might } \alpha \text{ and might } \beta$$

- Deontic modals: *R* may fail to be indisputable.
- Narrow scope free choice is robust.
- Wide scope free choice depends on whether the speaker is informed about the relevant permissions.

Different ways

There are different ways to implement neglect-zero.

- BSML^+: syntactic pragmatic enrichment via $[\cdot]^+$.
- BSML^*: model-theoretic implementation: the empty state is not among possible states.
- ne-free fragment of BSML (classical logic): explicit suspension of neglect-zero effects: zero-models may be retained when context demands them.

Predictions

effect	schematic form	BSML^+	BSML^*
positive FC	$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	✓	✓
dual prohibition	$\neg \Diamond (\alpha \vee \beta) \rightsquigarrow \neg \Diamond \alpha \wedge \neg \Diamond \beta$	✓	✓
negative FC	$\neg \Box (\alpha \wedge \beta) \rightsquigarrow \neg \Box \alpha \wedge \neg \Box \beta$	×	✓
negative conjunction	$\neg(\alpha \wedge \beta) \rightsquigarrow \Diamond_e \neg\alpha \wedge \Diamond_e \neg\beta$	×	✓

- One conjecture: BSML^+ captures robust detachable effects.
- BSML^* captures weaker non-detachable effects of globally omitting $\varnothing$.

Suspending Neglect-Zero

- Zero-models are cognitively costly, not impossible.
- Context can make empty witnesses salient.
- Mathematical and metalinguistic contexts often require attention to vacuity.
- The ne-free fragment of BSML, corresponding to classical logic, is a way to model contexts in which zero-models are explicitly kept available.

The empty operator $\varnothing$

$$M, s \vDash \oslash\phi \quad \text{iff} \quad M, s \vDash \phi \text{ or } s = \varnothing$$

$$M, s \dashv \oslash\phi \quad \text{iff} \quad M, s \dashv \phi$$

- ne makes emptiness visible.
- $\oslash$ is the corresponding emptiness-tolerating operator.
- It locally cancels non-emptiness requirements.
- Adding $\oslash$ to classical modal logic without ne does not increase expressive power: classical formulas already have the empty-state property.

Outline

Experimental Semantics

- Recall that ignorance is composed of two components

UNCERTAINTY	$\neg\Box_s A \wedge \neg\Box_s B$	speaker is not certain which
POSSIBILITY	$\Diamond_s A \wedge \Diamond_s B$	both disjuncts remain possible

- Competing theories differ in the way these two components are derived
- Experiments let us build contexts where these two components come apart.

The standard implicature view

- Start with a plain disjunction: $\Box_s(A \vee B)$
- The relevant stronger alternatives are roughly: $\Box_s A$ and $\Box_s B$
- By Quantity, the speaker is taken not to believe either stronger alternative:

$$\neg \Box_s A \wedge \neg \Box_s B \qquad \text{UNCERTAINTY}$$

- Given Quality, these upper-bound inferences yield lower-bound inferences:

$$\Diamond_s A \wedge \Diamond_s B \qquad \text{POSSIBILITY}$$

- Clear prediction: no possibility without uncertainty

Mystery box paradigm: separating the inferences

- Participants see three open boxes and one covered box.
- They are told that the covered box has the same contents as one of the open boxes.
- They judge whether a character's utterance is *Good* or *Bad*.

The mystery box contains a yellow ball or a blue ball.

	open boxes			hidden	UNCERTAINTY	POSSIBILITY
TARGET-1	A	AB	A	?	false	true
TARGET-2	A	AC	A	?	false	false

- In both targets, the speaker is certain that the box contains A.
- Only in TARGET-1 does B remain possible.

Results

condition	inference profile	acceptance
TARGET-1	uncertainty false, possibility true	95.2%
TARGET-2	uncertainty false, possibility false	44.2%

- Acceptance was very high in TARGET-1, but substantially lower in TARGET-2.
- On the standard implicature view, the possibility component is derived through the uncertainty component. It therefore predicts that possibility should not arise independently of uncertainty.
- TARGET-1 challenges this prediction: the possibility component remains available even though the uncertainty component is absent.
- Neglect-zero predicts exactly this weak ignorance profile:

$$[A \vee B]^+ \vDash \Diamond_s A \wedge \Diamond_s B \qquad [A \vee B]^+ \nvDash \neg \Box_s A \wedge \neg \Box_s B$$

Outline

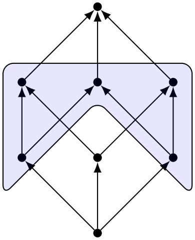
The Convexity Closure Property

Recall:

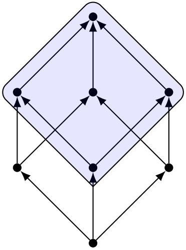
- Formulas of dependence logic are downward closed and have the empty team property. They need not be union closed.
- Interpreting teams as information states, downward closure corresponds to persistency: *once established, a fact remains established when we learn more.*

The nonemptiness atom ne breaks downward closure and the empty team property, but formulas of BSML are union closed, and also conform to another closure property which can be thought of as generalizing downward closure: convexity.

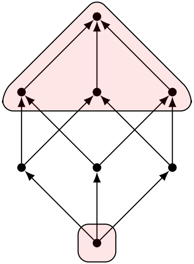
A formula ϕ is *convex* if $T \vDash \phi$, $S \vDash \phi$, and $S \subseteq U \subseteq T$ imply $U \vDash \phi$.



Convex



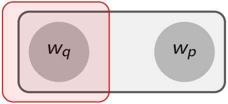
Convex



Not convex

Exercise

- We will soon prove that BSML is convex.
- However, BSML with the addition of $\oslash$ is not. Can you see why?
- $M, s \vDash \oslash\phi$ iff $M, s \vDash \phi$ or $s = \varnothing$
- Hint: consider the formula ϕ: $(p \wedge \text{ne}) \vee \top$



The team-semantic notion of convexity is derived from a more general notion of gaplessness in meanings.

There have been many proposals to the effect that such an absence of gaps constitutes a linguistic or cognitive universal of some kind.

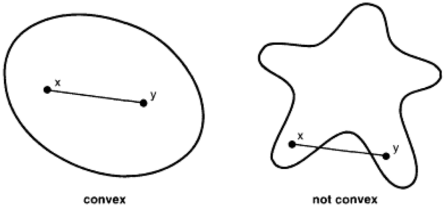


Image from Gärdenfors, The Geometry of Meaning: Semantics Based on Conceptual Spaces, 2000

Barwise & Cooper, 1981:

> *The simple NP's of any natural language express monotone quantifiers or conjunctions of monotone quantifiers.*

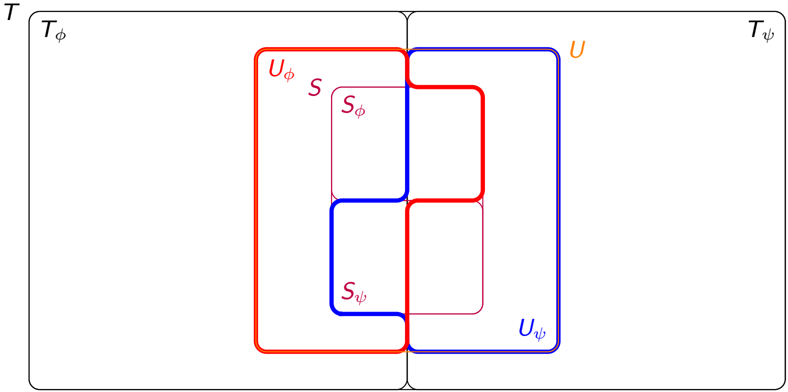
Gärdenfors, 2000:

> *A central feature of our cognitive mechanisms is that we assign properties to the objects that we observe [...] I primarily want to pin down the properties that are, in a sense, natural to our way of thinking [...] The third and most powerful criterion of a region is the following, which also relies on betweenness: A subset C of a conceptual space S is said to be convex if, for all points x and y in C, all points between x and y are also in C.*

One shows that BSML is convex by induction on the complexity of formulas.

Let us show that $\phi \vee \psi$ is convex, assuming that ϕ and ψ are union closed and convex

Let $T \vDash \phi \vee \psi$, $S \vDash \phi \vee \psi$, and $S \subseteq U \subseteq T$. Then $T = T_\phi \cup T_\psi$ and $S = S_\phi \cup S_\psi$, where $T_\phi \vDash \phi$, etc. By union closure, $T_\phi \cup S_\phi \vDash \phi$ and $T_\psi \cup S_\psi \vDash \psi$. Let $U_\phi := (T_\phi \cup S_\phi) \cap U$ and $U_\psi := (T_\psi \cup S_\psi) \cap U$. We have $S_\phi \subseteq U_\phi \subseteq T_\phi \cup S_\phi$, so $U_\phi \vDash \phi$ by convexity. Similarly $U_\psi \vDash \psi$. And $U = U_\phi \cup U_\psi$ so $U \vDash \phi \vee \psi$.



Closure Properties: Expressive Completeness

In the propositional setting, each formula ϕ defines or expresses a set of teams $\|\phi\| := \{T \mid T \vDash \phi\}$. Similarly in the modal setting, each formula defines a class of pointed models (M, T), where M is a model and T is a team.

The closure properties can be extended to these sets. E.g., a set of teams $\mathcal{P}$ is downward closed if $T \in \mathcal{P}$ and $S \subseteq T$ imply $S \in \mathcal{P}$.

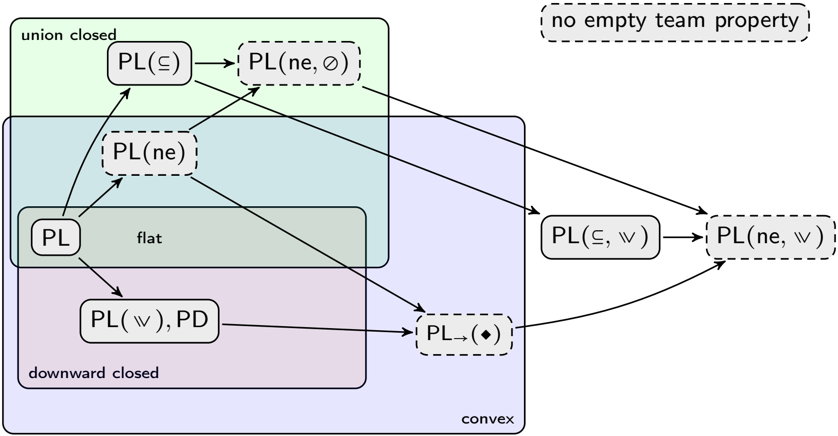
We say that a logic is *expressively complete* for a class of sets of teams if the class consists precisely of those definable by formulas of the logic.

For instance, propositional dependence logic PD is expressively complete for all downward-closed and empty-team-closed ($\varnothing \in \mathcal{P}$) sets. Intuitively, PD can express all downward-closed and empty-team-closed meanings.

Similarly, one can show in the modal setting that BSML can express all convex and union-closed meanings (that are also invariant under bounded bisimulation); BSML with $\oslash$, all union-closed meanings.

Expressive Completeness Results in the Literature

Some expressive completeness results in the propositional setting:



In the propositional setting, we characterize convex union-closed properties in the logic PL(ne) as follows:

Characteristic formulas for valuations and teams:

$$\chi_v := \bigwedge\{p \mid v \models p\} \land \bigwedge\{\neg p \mid v \not\models p\} \qquad \chi_s := \bigvee_{v \in s} \chi_v$$

$$w \models \chi_v \iff w = v \qquad t \models \chi_s \iff t \subseteq s$$

Characteristic formulas for flat (downward- and union-closed) sets:

$$t \models \bigvee_{s \in \mathcal{P}} \chi_s \iff t \subseteq \bigcup \mathcal{P} \iff t \in \mathcal{P} \text{ if } \mathcal{P} \text{ flat}$$

Characteristic formulas for upward-closed sets (here $\blacklozenge\phi := (\phi \land \text{ne}) \lor \top$):

$$t \models \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \lor \ldots \lor \chi_{v_n}) \iff \exists s \in \mathcal{P} = \{t_1, \ldots, t_n\} : s \subseteq t$$

$$\iff t \in \mathcal{P} \text{ if } \mathcal{P} \text{ upward closed}$$

Characteristic formulas for convex union-closed sets:

$$t \models \bigvee_{v \in s} \chi_v \land \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \lor \ldots \lor \chi_{v_n}) \iff \exists s \in \mathcal{P} = \{t_1, \ldots, t_n\} : s \subseteq t \text{ and } t \subseteq \bigcup \mathcal{P}$$

$$\iff t \in \mathcal{P} \text{ if } \mathcal{P} \text{ convex and union closed}$$

Similarly, $\mathrm{PL}(\mathrm{ne}, \oslash)$ captures all union-closed meanings:

Characteristic formulas for valuations and teams:

$$\chi_v := \bigwedge\{p \mid v \vDash p\} \wedge \bigwedge\{\neg p \mid v \nvDash p\} \qquad \theta_s := \bigvee_{v \in s}(\chi_v \wedge \mathrm{ne})$$

$$w \vDash \chi_v \iff w = v \qquad t \vDash \theta_s \iff t = s$$

Characteristic formulas for union- and empty-team-closed ($\varnothing \in \mathcal{P}$) sets $\mathcal{P}$:

$$t \vDash \bigvee_{s \in \mathcal{P}} \oslash \theta_s \iff t \in \mathcal{P} \text{ if } \mathcal{P} \text{ is union- and empty-team-closed}$$

Characteristic formulas for union-closed sets $\mathcal{P}$ with $\varnothing \notin \mathcal{P}$:

$$t \vDash \mathrm{ne} \wedge \bigvee_{s \in \mathcal{P}} \oslash \theta_s \iff t \in \mathcal{P} \text{ if } \mathcal{P} \text{ is union-closed and } \varnothing \notin \mathcal{P}$$

Exercise 1: two notions of $\bot$

Define $\bot_1 := \mathrm{ne} \wedge \neg\mathrm{ne}$ and $\bot_2 := p \wedge \neg p$.

1. Show that $\bot_1 \not\equiv \bot_2$, although they have the same behaviour on every nonempty state.
2. Is agreement on all nonempty states preserved when they occur as subformulas of a larger formula?
3. Compare their consequence sets. Which notion of contradiction validates explosion?
4. Let φ be a ne-free formula. Suppose that no nonempty team supports φ. Show that every such φ is equivalent to $\bot_2$.
5. Let $C(\varphi) := \varphi \wedge \neg\varphi$ and $C^{n+1}(\varphi) := C(C^n(\varphi))$.
   - 5.1 Determine $C^n(p)$ and $C^n(\mathrm{ne})$ for every $n \geq 1$.
   - 5.2 Explain why the sequence 'stabilizes' after one step for p, but only after two steps for ne.

Exercise 2: bilateral negation

For each of the following items, find a BSML-formula ϕ that satisfies the conditions listed:

1. $t \vDash \phi$ iff $t = \varnothing$, and $t \vDash \neg\phi$ iff $t \neq \varnothing$.
2. $t \vDash \phi$ iff $t = \varnothing$, and $t \vDash \neg\phi$ always holds.
3. $t \vDash \phi$ is never the case, and $t \vDash \neg\phi$ always holds.
4. $t \vDash \phi$ never holds, and $t \vDash \neg\phi$ iff $t \neq \varnothing$.
5. Neither $t \vDash \phi$ nor $t \vDash \neg\phi$ ever holds.
6. $t \vDash \phi$ never holds, and $t \vDash \neg\phi$ iff $t = \varnothing$.
7. $\phi \equiv p$, and $t \vDash \neg\phi$ never holds.
8. $t \vDash \phi$ iff $t \vDash \neg\phi$ iff $t = \varnothing$. (Hint: this requires the modal operators)
9. (Hard) $\phi \equiv p$ and $\neg\phi \equiv \neg p \wedge \Diamond q$. (Hint: make use of ϕ as in item 6)

Show that for all BSML-formulas ϕ, $s \vDash \phi$ and $t \vDash \neg\phi$ implies $s \cap t = \varnothing$.

Exercise 3: convexity

For each of the following formulas or properties, determine whether it is convex. If not convex, give a counterexample.

1. $\mathrm{dep}(p, q)$
2. $p \mathbin{\veebar} q$
3. $p \mathbin{\veebar} ((q \wedge \mathrm{ne}) \vee \top)$
4. $(p \mathbin{\veebar} \neg p) \wedge ((q \wedge \mathrm{ne}) \vee \top)$
5. For each $w \in t$, there is $v \in t$ such that $w(p) = v(q)$
6. $\exists x P y$
7. $\exists x((Py \wedge \mathrm{ne}) \vee \top)$

Find convex formulas ϕ, ψ (using any connectives we have introduced) such that $\phi \vee \psi$ is not convex.