Lecture 3: Questions

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Team Semantics: Linguistic and Philosophical Applications
ESSLLI 2026, Prague

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Readings

- Ciardelli, I. (2022). *Inquisitive Logic: Consequence and Inference in the Realm of Questions*.
  https://doi.org/10.1007/978-3-031-09706-5
- Ciardelli, I., Groenendijk, J., & Roelofsen, F. (2018). *Inquisitive Semantics*.
  https://doi.org/10.1093/oso/9780198814788.001.0001
- Ciardelli, I. & Roelofsen, F. (2015). "Inquisitive dynamic epistemic logic." *Synthese*, 192, 1643—1687.
  https://doi.org/10.1007/s11229-014-0404-7

Outline

1. Questions and Resolutions
2. Basic Inquisitive Semantics
3. Inquisitive Epistemic Logic

Three Perspectives on Questions

- **Syntax:** interrogative clauses display characteristic formal features: word order, wh-phrases, particles, intonation.
- **Pragmatics:** asking a question is an act: a request to provide information.
- **Semantics:** a question is the abstract content associated with an interrogative, and also the content embedded under verbs like *know* and *wonder*.

The Truth-Conditional Challenge

Classical possible-world semantics gives declaratives truth conditions:

$$|\varphi|_M^C = \{w \in W \mid M, w \models \varphi\}$$

- This works well for assertions such as *Aicha is coming*.
- But interrogatives do not have ordinary truth conditions:

(1) a. Is Aicha coming?
 b. Who is coming?

- Embedded cases show that declaratives and interrogatives cannot simply have the same content:

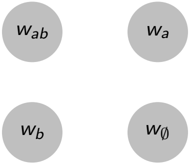
(2) a. Bo knows that Aicha is coming
 b. Bo knows whether Aicha is coming

The Inquisitive Shift

- The answerhood view identifies the meaning of an interrogative with conditions on what counts as an answer.
- Such meanings are typically represented as sets of propositions.
- The resolution view identifies the meaning of an interrogative with conditions on what information resolves the issue it raises.
- Such meanings are represented as sets of information states.
- In inquisitive semantics, question meaning is therefore understood as resolution conditions.

Running Logical Space

Let a mean that Aicha is coming and b that Bo is coming.



- w_{ab}: both Aicha and Bo are coming.
- w_a: only Aicha is coming.
- w_b: only Bo is coming.
- $w_\emptyset$: neither is coming.

Information states are subsets of this logical space.

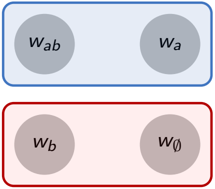
Polar Question: Is Aicha Coming?

Recall the global or inquisitive disjunction:

$$M, s \models \phi \mathbin{\mathbb{V}} \psi \quad \text{iff} \quad M, s \models \phi \text{ or } M, s \models \psi$$

We use this to formalize questions:

Is Aicha coming? $\quad a \mathbin{\mathbb{V}} \neg a$



$\{\{w_{ab}, w_a\}, \{w_b, w_\emptyset\}\}$

$\{w_{ab}, w_a\}$, $\{w_b, w_\emptyset\}$,
$\{w_{ab}\}, \{w_a\}, \{w_b\}, \{w_\emptyset\}, \emptyset$

- every substate of a resolving state resolves;
- e.g. $\{w_a, w_b\}$ does not resolve.

Interrogative Entailment

A natural entailment notion for questions:

Question Q_1 entails question Q_2 iff every state resolving Q_1 also resolves Q_2.

$$[Q]_M = \{s \mid s \text{ resolves } Q_1\}$$

$$Q_1 \models Q_2 \quad \iff \quad [Q_1]_M \subseteq [Q_2]_M.$$

Who is coming? $\models$ *Is Aicha coming?*

Resolving exactly who is coming is enough information to resolve whether Aicha is coming.

Generalized Entailment

The same inclusion notion applies across sentence types.

1. declarative to declarative:

$$\text{Aicha and Bo are coming} \models \text{Aicha is coming;}$$

2. interrogative to interrogative:

$$\text{Who is coming?} \models \text{Is Aicha coming?;}$$

3. declarative to interrogative:

$$\text{Only Aicha is coming} \models \text{Who is coming?}$$

when the declarative provides enough resolving information.

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From Worlds to Information States

Classical semantics evaluates formulas at worlds:

$$M, w \models \varphi$$

Inquisitive semantics evaluates formulas at information states:

$$M, s \models \varphi \qquad (s \subseteq W)$$

- s represents the live possibilities compatible with the available information.
- If $t \subseteq s$, we say t is an enhancement of s—t contains more information.
- Supporting a formula means that the information in s is enough for the semantic task associated with the formula.
- For declaratives this task is truth. For interrogatives it is resolution.

Inquisitive Propositions

A classical proposition is a set of worlds:

$$|\varphi|_M^C = \{w \in W \mid M, w \models \varphi\}$$

An inquisitive proposition is a set of information states:

$$|\varphi|_M^I = \{s \subseteq W \mid M, s \models \varphi\}$$

Issues

An **issue** over W is a set $I \subseteq \mathcal{P}(W)$ that is:

1. downward closed: if $s \in I$ and $t \subseteq s$, then $t \in I$;
2. non-empty: $\emptyset \in I$.

- Intuitively, $s \in I$ means that s resolves the issue.
- Downward closure: if some information resolves an issue, then stronger information also resolves it.
- The empty state resolves every issue vacuously.

Alternatives

The **alternatives** of an inquisitive proposition Π are its maximal elements:

$$\mathsf{Alt}(\Pi) = \{s \in \Pi \mid \text{there is no } t \in \Pi \text{ with } s \subsetneq t\}$$

- Alternatives are maximal ways of resolving the issue expressed by Π.
- Let $\mathsf{Alt}(\Pi)^{\downarrow} = \{s \subseteq W \mid s \subseteq t \text{ for some } t \in \mathsf{Alt}(\Pi)\}$.
- By downward closure,

$$\mathsf{Alt}(\Pi)^{\downarrow} \subseteq \Pi$$

- We say that Π is normal if in addition $\Pi \subseteq \mathsf{Alt}(\Pi)^{\downarrow}$. That is Π is generated by its alternatives:

$$\Pi = \mathsf{Alt}(\Pi)^{\downarrow}$$

- We will only be concerned with settings in which propositions are normal.

Informative Content

The informative content of an inquisitive proposition is:

$$\mathrm{info}(\Pi) = \bigcup \Pi$$

- It is the set of worlds compatible with the information conveyed by Π.
- Π is **informative** iff $\mathrm{info}(\Pi) \neq W$.
- Π is **non-informative** iff $\mathrm{info}(\Pi) = W$.
- If Π represents a statement, $\mathrm{info}(\Pi)$ is simply the information conveyed by the statement. If Π represents a question, $\mathrm{info}(\Pi)$ is the information assumed or presupposed by the question.

Inquisitiveness

A proposition Π is **non-inquisitive** iff it contains all substates of its informative content:

$$\Pi = \mathcal{P}(\mathsf{info}(\Pi))$$

Equivalently, in finite models, Π has a single alternative.

- Non-inquisitive propositions behave like classical propositions.
- Inquisitive propositions have more than one maximal way of being resolved.

Basic Propositional Inquisitive Logic InqB

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \mathbin{\mathbb{V}} \phi \qquad (p \in \mathsf{Prop})$$

Let $M = (W, V)$ be a propositional model, where $V(p) \subseteq W$, and let $s \subseteq W$ be an information state. Support is defined recursively as follows:

$$\begin{aligned}
M, s \models p &\iff s \subseteq V(p) \\
M, s \models \bot &\iff s = \emptyset \\
M, s \models \varphi \wedge \psi &\iff M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models \varphi \to \psi &\iff \text{for every } t \subseteq s,\ M, t \models \varphi \text{ implies } M, t \models \psi \\
M, s \models \varphi \mathbin{\mathbb{V}} \psi &\iff M, s \models \varphi \text{ or } M, s \models \psi
\end{aligned}$$

- $\mathbb{V}$ is inquisitive disjunction. It introduces alternative support conditions rather than splitting the team.
- $\neg\phi := \phi \to \bot$

Support and Truth

We write simply $M, w \models \phi$ for $M, \{w\} \models \phi$.

Truth conditions are recovered as a special case of support conditions: ϕ is said to be true in a world w in a model M just in case $M, w \models \phi$.

Support set of ϕ:

$$[\varphi]_M = \{s \subseteq W \mid M, s \models \varphi\}$$

Truth set of ϕ:

$$|\varphi|_M = \{w \in W \mid M, w \models \varphi\}$$

By downward closure,

$$|\phi|_M = \bigcup [\phi]_M.$$

I.e., $|\phi|_M$ is the informative content of the issue $[\phi]_M$.

Statements and Questions

ϕ is said to be *truth-conditional* (flat) or a *statement* if its truth conditions determine its support conditions:

$$M, s \models \phi \iff M, w \models \phi \text{ for all } w \in s$$

Equivalently: for all M, the issue $[\phi]_M$ is non-inquisitive: $[\phi]_M = \mathcal{P}(|\phi|_M)$.

If ϕ is not truth-conditional, it is a *question*.

- Classical ($\mathbin{\vee\!\!\!\vee}$-free) formulas are truth-conditional.
- Conversely, every truth-conditional InqB-formula is equivalent to a classical formula.
- Adding inquisitive disjunction extends propositional logic with questions, but does not create new truth-conditional propositional statements.

Question Formation

The polar question whether φ is defined as:

$$?\varphi := \varphi \mathbin{\mathbb{V}} \neg\varphi$$

Thus:

$$M, s \models ?\varphi \quad \Longleftrightarrow \quad M, s \models \varphi \text{ or } M, s \models \neg\varphi$$

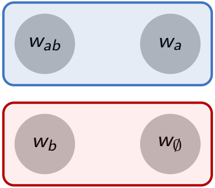
- A state supports $?\varphi$ iff it settles φ one way or the other.
- If φ is a statement, the alternatives of $?\varphi$ are its positive and negative truth sets.

Example: $?a$

Q: Is Aicha coming?

A1: Yes.

A2: No.

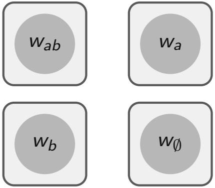


$$[?a]_M = \mathcal{P}(\{w_{ab}, w_a\}) \cup \mathcal{P}(\{w_b, w_\emptyset\})$$

- $\{w_{ab}, w_a\}$ supports a.
- $\{w_b, w_\emptyset\}$ supports $\neg a$.
- $\{w_a, w_b\}$ supports neither. It leaves the issue open.

Example: $?a \wedge ?b$

Q: Is Aicha coming and is Bo coming?



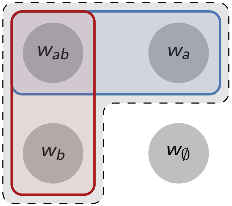
$$[?a \wedge ?b]_M = \{\{w_{ab}\}, \{w_a\}, \{w_b\}, \{w_\emptyset\}\}$$

Presuppositions

What does it mean to say that a question ϕ is true at w? Given $|\phi|_M = \bigcup[\phi]_M$, the worlds at which ϕ is true are those belonging to some state where ϕ is resolved. So ϕ is true when it can be truthfully resolved.

We can associate with each formula ϕ a classical formula α_ϕ with the same truth-conditions ($|\phi|_M = |\alpha_\phi|_M$) by replacing each $\vee\!\!\!\vee$ with $\vee$. We can think of α_ϕ as the *presupposition* of ϕ: in order for a question ϕ to be truthfully resolved, the statement α_ϕ must be true. Then we may also say: ϕ is true just in case its presupposition α_ϕ is true.

Example: Presupposition



$$[a \mathbin{\vee\!\!\!\vee} b]_M = \mathcal{P}(\{w_{ab}, w_a\}) \cup \mathcal{P}(\{w_{ab}, w_b\})$$

$$[a \vee b]_M = \mathcal{P}(\{w_{ab}, w_a, w_b\})$$

$$|a \mathbin{\vee\!\!\!\vee} b|_M = |a \vee b|_M = \{w_{ab}, w_a, w_b\}$$

Why Implication Quantifies over Substates

$$M, s \models \varphi \to \psi \quad \Longleftrightarrow \quad \forall t \subseteq s \, (M, t \models \varphi \Rightarrow M, t \models \psi)$$

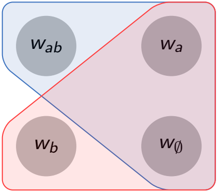
- Implication internalizes entailment relative to the information available in s.
- It says: every possible enhancement of s that supports the antecedent supports the consequent.
- This generalizes the Ramsey-test idea from statements to questions:

  To interpret a conditional $\phi \to \psi$ in a belief state, hypothetically assume that ϕ is true [that there is enough information to resolve ϕ], then accept $\phi \to \psi$ just in case ψ is true [there is enough information to resolve ψ] in the resulting state.

Example: $b \rightarrow ?a$

Q: If Bo is coming, is Aicha coming?

This is supported in a state s in a model M just in case $s \cap |b|_M$ supports $?a$: if an information state is enhanced with the information that b it determines whether a.



$$[b \rightarrow ?a]_M =$$

$$\mathcal{P}(\{w_{ab}, w_a, w_\emptyset\}) \cup \mathcal{P}(\{w_a, w_b, w_\emptyset\})$$

Equivalently, s supports $b \rightarrow a$, or it supports $b \rightarrow \neg a$. This corresponds to the following two answers to Q:

A1: Yes, if Bo is coming, Aicha is coming.

A2: No, if Bo is coming, Aicha is not coming.

Inquisitive Entailment

For formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \models \varphi \quad \iff \quad \text{for all } M, s:\; M, s \models \Gamma \Rightarrow M, s \models \varphi$$

Equivalently, for a single premise:

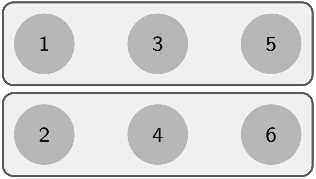
$$\varphi \models \psi \quad \iff \quad [\varphi]_M \subseteq [\psi]_M \text{ in every } M$$

- Declarative entailment is recovered as a special case.
- Question entailment becomes entailment between information types.

Questions as Information Types

The unique alternative $|\alpha|_M$ for a statement α can be viewed as a piece of information: the information that α is true.

Questions have multiple alternatives. Consider the outcomes of a die roll, and the question parity as to whether or not the outcome is even:



In this model, parity has the alternatives:

- $\{w_1, w_3, w_5\} = |\text{odd}|_M$
- $\{w_2, w_4, w_6\} = |\text{even}|_M$

These correspond to distinct pieces of information about the parity of the outcome. We think of them as collectively instantiating the information type parity. To settle the question is to establish some information of this type.

Localizing Entailment

We previously defined global entailment. We can also define a variant with a fixed model M and state s.

$$\Gamma \models_s^M \varphi \quad \Longleftrightarrow \quad \forall t \subseteq s \left(M, t \models \Gamma \Rightarrow M, t \models \varphi\right).$$

- The substates $t \subseteq s$ represent possible refinements of the information currently available.
- Thus, $\Gamma \models_s^M \varphi$ says that every refinement of s which supports Γ also supports φ.
- We also write $\Gamma \models_M \phi$ for $\Gamma \models_W^M$, where W is the universe of the model M.

Entailment as Relation Among Information Types

Entailment between ϕ and ψ holds if any piece of information of type ϕ implies some type of information of type ψ:

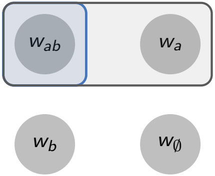
$$\phi \models_M \psi \quad \text{iff} \quad \text{for each alternative } a \text{ of } [\phi]_M \text{ there is an alternative } a' \text{ of } [\psi]_M \text{ with } a \subseteq a'$$

Declarative to Declarative Entailment

Relation between pieces of information:

$$\alpha \models_M \beta \iff |\alpha|_M \subseteq |\beta|_M$$

Aicha and Bo are coming $\models$ Aicha is coming;

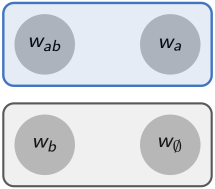


Declarative to Interrogative Entailment

The information that α is true yields some piece of information of type μ:

$$\alpha \models_M \mu \iff |\alpha|_M \subseteq a \text{ for some alternative } a \text{ of } \mu$$

Aicha is coming $\models$ Is Aicha coming?

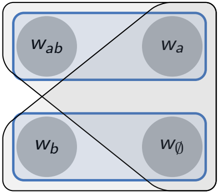


Interrogative to Interrogative Entailment

Any piece of information of type μ implies some type of information of type ν:

$$\mu \models_M \nu \quad \text{iff} \quad \begin{array}{l}\text{for each alternative } a \text{ of } [\mu]_M \text{ there is} \\ \text{an alternative } a' \text{ of } [\nu]_M \text{ with } a \subseteq a'\end{array}$$

Is Aicha coming? $\models$ If Bo is coming, is Aicha coming?

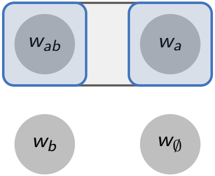


Interrogative to Declarative Entailment

Any information of type μ yields the information that α is true:

$$\mu \models_M \alpha \iff a \subseteq |\alpha|_M \text{ for all alternatives } a \text{ of } [\mu]_M$$
$$\iff |\alpha_\mu|_M \subseteq |\alpha|_M \text{ for } \alpha_\mu \text{ the presupposition of } \mu$$

Are both Bo and Aicha coming, or only Aicha? $\models$ Aicha is coming



Q-Dependency

Let $\mu_1, \ldots, \mu_n$ and ν be questions. We say that ν depends on $\mu_1, \ldots, \mu_n$ in M, s iff:

$$\mu_1, \ldots, \mu_n \models_s^M \nu$$

Equivalently:

$$\forall t \subseteq s \, (M, t \models \mu_1 \text{ and } \cdots \text{ and } M, t \models \mu_n \Rightarrow M, t \models \nu)$$

- Resolving $\mu_1, \ldots, \mu_n$ is sufficient, relative to s, to resolve ν.
- The questions $\mu_1, \ldots, \mu_n$ are the determining issues.
- The question ν is the determined issue.

Example: Conditional Q-Dependency

Let a mean that Alice is coming and b that Bo is coming. Suppose the background information says that exactly one of them is coming:

$$\alpha_{\oplus} := (a \wedge \neg b) \vee^{*} (\neg a \wedge b)$$

Relative to this background, settling whether Alice is coming settles whether Bo is coming:

$$\alpha_{\oplus}, ?a \models^{M}_{s} ?b$$

- If a is true under $\alpha_{\oplus}$, then b is false.
- If a is false under $\alpha_{\oplus}$, then b is true.
- Hence every refinement of s that supports $\alpha_{\oplus}$ and resolves $?a$ also resolves $?b$.

Conditional Dependencies

The previous example has the general form:

$$\alpha, \mu \models_s^M \nu$$

where α is background information and μ, ν are questions.

Equivalently, for flat α:

$$\mu \models_s^{M,\alpha} \nu \quad \iff \quad \forall t \subseteq s \cap |\alpha|_M \, (M, t \models \mu \Rightarrow M, t \models \nu)$$

- Declarative premises restrict the relevant possibilities.
- Question premises specify which information is taken as input.
- The conclusion specifies which issue is thereby resolved.

Connection with Dependence Logic

The same idea recovers functional dependence in team semantics.

For a team X of assignments:

$$M, X \models \mathrm{dep}(\vec{x}, y) \iff \forall g, h \in X\,(g(\vec{x}) = h(\vec{x}) \Rightarrow g(y) = h(y))$$

Let $\lambda_{\vec{x}}$ be the question asking for the value of $\vec{x}$, and let λ_y be the question asking for the value of y. Then:

$$M, X \models \mathrm{dep}(\vec{x}, y) \iff \lambda_{\vec{x}} \models^M_X \lambda_y$$

- Any subteam that determines $\vec{x}$ also determines y.
- Thus, functional dependence is a special case of question entailment.

Examples of Q-Dependencies in a Team

X	x	y
g_1	1	0
g_2	1	2
g_3	2	1
g_4	2	3
g_5	5	4
g_6	5	6

- The value of y determines the value of x in X.
  $M, X \models \mathsf{dep}(y, x) \qquad \lambda_y \models^M_X \lambda_x$
- The value of x determines the parity of y (whether y is even or odd). $\quad \lambda_y \models^M_X ?\mathrm{Even}(y)$
- The parity of x determines the parity of y.
  $?\mathrm{Even}(x) \models^M_X ?\mathrm{Even}(y)$
- The value of x and whether $x < y$ determines the value of y.
  $\lambda_x, ?(x < y) \models^M_X \lambda_y$

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Standard Epistemic Logic

Recall that standard epistemic logic is designed to describe and reason about agents' knowledge. It has modal operators K_a, where $K_a\phi$ is read as "agent *a* knows that ϕ".

Interpreted on epistemic models $M = (W, V, \sigma_{\mathcal{A}})$ for a set of agents $\mathcal{A}$, where $\sigma_{\mathcal{A}} = \{\sigma_a \mid a \in \mathcal{A}\}$ is a set of epistemic maps $\sigma_a : W \to \mathcal{P}(W)$ that assign an information state $\sigma_a(w)$ to any world w, satisfying:

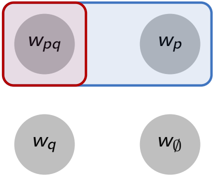
- Factivity: $w \in \sigma_a(w)$
- Introspection: if $v \in \sigma_a(w)$, then $\sigma_a(w) = \sigma_a(v)$

The state $\sigma_a(w)$ represents what agent *a* knows at w. The factivity condition requires that the information $\sigma_a(w)$ be truthful. The introspection condition requires that agents know their knowledge state: if the state of *a* at *v* differs from their state in *w*, they can tell these worlds apart.

$$
\begin{aligned}
M, w \models K_a\phi \iff& \text{ for any } v \in \sigma_a(w), M, v \models \phi \\
\iff& \ \sigma_a(w) \subseteq |\phi|_M
\end{aligned}
$$

Examples

$$M, w \models K_a\phi \iff \text{for any } v \in \sigma_a(w), M, v \vDash \phi$$
$$\iff \sigma_a(w) \subseteq |\phi|_M$$



$\sigma_a(w_{pq}) = \{w_{pq}\}$,
$\sigma_b(w_{pq}) = \sigma_b(w_p) = \{w_{pq}, w_p\}$

$|p \wedge q|_M = \{w_{pq}\}$
$|K_b p|_M = \{w_{pq}, w_p\}$

In w_{pq}:

- a knows that p and q: $M, w_{pq} \models K_a(p \wedge q)$
- b knows that p, and does not know that q: $M, w_{pq} \models K_b p \wedge \neg K_b q$
- a knows that b knows that p: $M, w_{pq} \models K_a K_b p$
- b does not know that a knows that q: $M, w_{pq} \models \neg K_b K_a q$

Common Knowledge

A sentence ϕ is common knowledge if every agent a knows that ϕ, every agent b knows that every agent a knows that ϕ, etc.

$$M, w \models K_* \phi \iff M, w \models K_{a_1} K_{a_2} \dots K_{a_n} \phi \text{ for all } a_1, \dots, s_n \in \mathcal{A}$$

This can also be specified with a common knowledge epistemic map:

$$\sigma_*(w) = \{v \mid \exists u_0, \dots, u_{n+1} \in W, a_0, \dots a_n \in \mathcal{A} : \\ u_0 = w, u_{n+1} = v, \forall i \leq n : u_{i+1} \in \sigma_{a_i}(u_i)\}$$

Dynamic Epistemic Logic

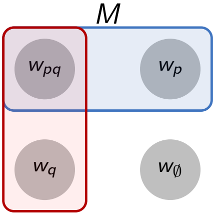
Dynamic epistemic logic additionally models how states of affairs change when actions are performed.

We consider only type of action: public announcement of a sentence ϕ. We assume such public announcements are always truthful, and have the effect of making it common knowledge that ϕ was true at the moment of utterance.

A public announcement of ϕ transforms a model $M = (W, V, \sigma_{\mathcal{A}})$ into a model $M^\phi = (W^\phi, V^\phi, \sigma^\phi_{\mathcal{A}})$ obtained by restricting M to the worlds in which ϕ is true. I.e.,

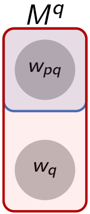
- $W^\phi = W \cap |\phi|_M$
- $V^\phi = V \restriction_{W^\phi}$
- $\sigma^\phi_{\mathcal{A}} = \{\sigma^\phi_a \mid a \in \mathcal{A}\}$ where $\sigma^\phi_a(w) = \sigma_a(w) \cap |\phi|_M$

If ϕ is *factive* (does not contain epistemic operators), then $K_*\phi$ holds after an announcement of ϕ. However, if ϕ is not factive, an announcement of ϕ does not guarantee common knowledge and may even make ϕ false:

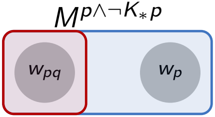


$$\sigma_a(w_{pq}) = \sigma_a(w_q) = \{w_{pq}, w_q\}$$
$$\sigma_b(w_{pq}) = \sigma_b(w_p) = \{w_{pq}, w_p\}$$

$$|q|_M = \{w_{pq}, w_q\}$$
$$|p \wedge \neg K_* p|_M = \{w_{pq}, w_p\}$$



$$M^q, w_{pq} \models K_* q$$

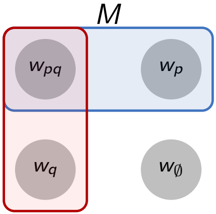


$$M^{p\wedge\neg K_* p}, w_{pq} \models K_* p$$
$$M^{p\wedge\neg K_* p}, w_{pq} \not\models p \wedge \neg K_* p$$

Public Announcement Operator

The language of standard epistemic logic can be extended with a dynamic modality $[\phi]$ for talking about what is the case after a public announcement of ϕ:

$$M, w \models [\phi]\psi \iff w \notin |\phi|_M \text{ or } M^\phi, w \models \psi$$

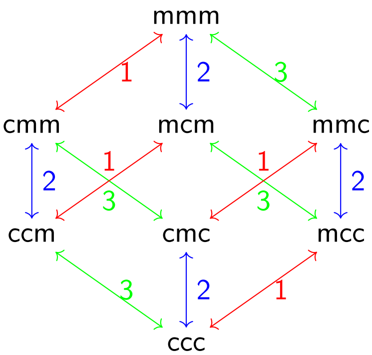


$$M, w_{pq} \models [q]K_* q$$
$$M, w_{pq} \models \neg[p \wedge K_* p]p \wedge K_* p$$

Muddy Children Puzzle

Three children have been playing in the mud. Their father summons them and says, "At least one of you has mud on your face." He then asks each of them: "Do you know whether you have mud on your face?" The children can see each others' faces, but may not check their own. They answer simultaneously, "no". The father asks the same question again, and some children answer, "no", while others answer, "yes". The father repeats the question a third time and now all children answer, "yes". How many children have mud on their faces?

M:



"At least one of you has mud on your face"

$\phi := m_1 \vee m_2 \vee m_3$

$|\phi|_M = W \setminus \{ccc\}$

M^ϕ:

$$\begin{tikzcd}
 & mmm \arrow[dl, leftrightarrow, "1"'] \arrow[d, leftrightarrow, "2"] \arrow[dr, leftrightarrow, "3"] & \\
cmm \arrow[d, leftrightarrow, "2"] \arrow[dr, leftrightarrow, "3"] & mcm \arrow[dl, leftrightarrow, "1"'] \arrow[dr, leftrightarrow, "3"] & mmc \arrow[dl, leftrightarrow, "1"'] \arrow[d, leftrightarrow, "2"] \\
ccm & cmc & mcc
\end{tikzcd}$$

All children respond: I don't know.

$\psi := \neg K_1 m_1 \wedge \neg K_1 \neg m_1 \wedge \neg K_2 m_2 \wedge \neg K_2 \neg m_2 \wedge \neg K_3 m_3 \wedge \neg K_3 \neg m_3$

$|K_1 m_1|_{M^\phi} = \{mcc\}$, $|K_2 m_2|_{M^\phi} = \{cmc\}$, $|K_2 m_2|_{M^\phi} = \{ccm\}$

$|\psi|_{M^\phi} = W^\phi \setminus \{ccm, cmc, mcc\}$

$(M^{\phi})^{\psi}$:

$$
\begin{tikzcd}
 & \mathrm{mmm} \arrow[dl, leftrightarrow, "1"'] \arrow[d, leftrightarrow, "2"] \arrow[dr, leftrightarrow, "3"] & \\
\mathrm{cmm} & \mathrm{mcm} & \mathrm{mmc}
\end{tikzcd}
$$

Some children respond: I know; others, I don't know. $(\neg\psi)$

$$|\neg\psi|_{M^{\psi}} = M^{\psi} \setminus \{mmm\}$$

$((M^{\phi})^{\psi})^{\neg\psi}$:

cmm mcm mmc

Let $\chi := (\neg m_1 \wedge m_2 \wedge m_3) \vee (m_1 \wedge \neg m_2 \wedge m_3) \vee (m_1 \wedge m_2 \wedge \neg m_3)$, i.e., "one child is clean".

We have $((M^{\phi})^{\psi})^{\neg\psi} \models \chi$, so $M \models [\phi][\psi][\neg\psi]\chi$.

Inquisitive Epistemic Logic

In inquisitive epistemic logic, we model not only agents' knowledge, but also the questions they would like to see resolved. These are captured together in the issues the agents *entertain*.

At each world w, each agent a is assigned an *inquisitive state* $\Sigma_a(w)$, modeled as an issue over their epistemic state $\sigma_a(w)$, i.e., $\sigma_a(w) = \bigcup \Sigma_a(w)$ This reflects the agent's desire to locate the actual world more precisely within their information state.

Denote by Π_s the set of issues I over an information state s ($\bigcup I = s$) and, given a model M with worlds W, let $\Pi = \bigcup_{s \subseteq W} \Pi_s$ be the set of issues over some state $s \subseteq W$.

An inquisitive epistemic model is a triple $M = (W, V, \Sigma_{\mathcal{A}})$, where $\Sigma_{\mathcal{A}} = \{\sigma_a \mid a \in \mathcal{A}\}$ is a set of state maps $\Sigma_a : W \to \Pi$ that assign an issue $\Sigma_a(w)$ to any world w such that for $\sigma_a(w) := \bigcup \Sigma_a(w)$:

- Factivity: $w \in \sigma_a(w)$
- Introspection: if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$

Inquisitive Epistemic Operators

We extend propositional inquisitive logic with operators $K_a\phi$ ("a knows that ϕ") and $E_a\phi$ ("a entertains ϕ").

$$\begin{aligned} M, s \models K_a\phi &\iff \text{for any } w \in s : M, \sigma_a(w) \vDash \phi \\ &\iff \text{for any } w \in s : \sigma_a(w) \in [\phi]_M \\ M, s \models E_a\phi &\iff \text{for any } w \in s \text{ and any } t \in \Sigma_a(w) : M, t \vDash \phi \\ &\iff \text{for any } w \in s : \Sigma_a(w) \subseteq [\phi]_M \end{aligned}$$

Note that these are both statements (i.e., they are truth-conditional).

For a statement α, $K_a\alpha$ is just the standard epistemic modality: $[\alpha]_M = \mathcal{P}(|\alpha|_M)$, so at a world w, $\sigma_a(w) \in [\alpha]_M$ iff $\sigma_a(w) \subseteq |\alpha|_M$.

For a question ϕ, $K_a\phi$ is a true at a world w if a has sufficient information to resolve ϕ at w. E.g., $K_a?p$ is true at w if $\sigma_a(w)$ supports either p or $\neg p$, i.e., a knows *whether* p holds.

Entertain Operator

$$\begin{aligned} M, s \models K_a\phi &\iff \text{for any } w \in s : M, \sigma_a(w) \vDash \phi \\ &\iff \text{for any } w \in s : \sigma_a(w) \in [\phi]_M \\ M, s \models E_a\phi &\iff \text{for any } w \in s \text{ and any } t \in \Sigma_a(w) : M, t \vDash \phi \\ &\iff \text{for any } w \in s : \Sigma_a(w) \subseteq [\phi]_M \end{aligned}$$

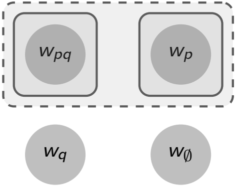
The notion of 'entertain' represented by E_a is a technical one. $E_a\phi$ may be thought of as "*a* knows ϕ or *a* wonders whether ϕ".

- $K_a\phi \models E_a\phi$ by downward closure since $\sigma_a(w) = \bigcup \Sigma_a(w)$.
- If α is a statement, $E_a\alpha \equiv K_a\alpha$ (if α is supported by every $t \in \Sigma_a(w)$, it is true at every $v \in \bigcup \Sigma_a(w) = \sigma_a(w)$, thus supported by $\sigma_a(w)$).
- If ϕ is a question, $E_a\phi$ expresses that resolving *a*'s issues entails resolving ϕ—*a* wants to see ϕ resolved. In case *a* already has enough information to resolve ϕ, we have $K_a\phi$, whence also $E_a\phi$. We can characterize the case in which *a* does not have enough information to resolve ϕ but wants to see it resolved as *a wondering* about ϕ:

$$W_a\phi := \neg K_a\phi \wedge E_a\phi$$

Examples

$$
\begin{aligned}
M, s \models K_a\phi &\iff \text{for any } w \in s : M, \sigma_a(w) \vDash \phi \\
&\iff \text{for any } w \in s : \sigma_a(w) \in [\phi]_M \\
M, s \models E_a\phi &\iff \text{for any } w \in s \text{ and any } t \in \Sigma_a(w) : M, t \vDash \phi \\
&\iff \text{for any } w \in s : \Sigma_a(w) \subseteq [\phi]_M
\end{aligned}
$$



$\Sigma_a(w_{pq}) = \{\{w_{pq}\}, \{w_p\}\}$
$\sigma_a(w_{pq}) = \{w_{pq}, w_p\}$

- a knows that p, and so knows whether p: $M, w_{pq} \models K_a p$ and $M, w_{pq} \models K_a?p$.
- Since a knows that p, a entertains p: $M, w_{pq} \models E_a p$.
- a does not know whether q, but does entertain $?q$, so a wonders whether q: $M, w_{pq} \models \neg K_a?q \wedge E_a?q$, so $M, w_{pq} \models W_a?q$

Public Entertainment

Analogously to common knowledge, say that a sentence ϕ is publicly entertained if every agent a entertains ϕ, every agent b knows that every agent a entertains ϕ, every agent c knows this, etc.

$$M, s \models E_* \phi \iff M, s \models K_{a_1} K_{a_2} \dots E_{a_n} \phi \text{ for all } a_1, \dots, s_n \in \mathcal{A}$$

This induces an associated map:

$$\Sigma_*(w) = \{s \mid \exists u_0, \dots, u_n \in W, a_0, \dots a_n \in \mathcal{A} : \\ u_0 = w, s \in \Sigma_{a_n}(v_n), \forall i < n : u_{i+1} \in \sigma_{a_i}(u_i)\}$$

We can also introduce a public wonder modality, expressing that the agents $\mathcal{A}$ publicly entertain a sentence that is not publicly settled:

$$W_* \phi := \neg K_* \phi \wedge E_* \phi$$

Note that $W_* \phi$ does not entail $W_a \phi$ for any agent a. It might for instance be the case that ϕ is publicly entertained, that each agent privately has enough information to resolve ϕ, but that this is the case is not common knowledge.

Inquisitive Dynamic Epistemic Logic

In addition to public announcements of statements, we now also model public announcements of questions. We assume a public announcement of a question ϕ raises a public issue: the effect of such an announcement is to make the issue expressed by ϕ (at the time of utterance) publicly entertained.

A public announcement of ϕ transforms a model $M = (W, V, \Sigma_{\mathcal{A}})$ into a model $M^\phi = (W^\phi, V^\phi, \Sigma^\phi_{\mathcal{A}})$ obtained by restricting M to the worlds in which ϕ is true and intersecting all inquisitive states with $[\phi]_M$. I.e.,

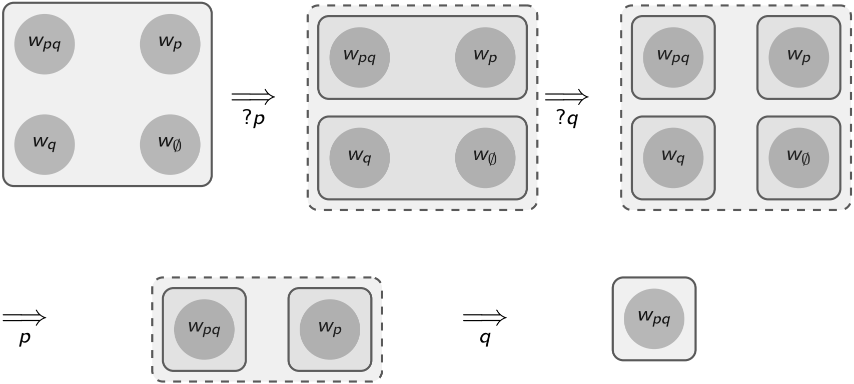
- $W^\phi = W \cap |\phi|_M$
- $V^\phi = V \restriction_{W^\phi}$
- $\Sigma^\phi_{\mathcal{A}} = \{\Sigma^\phi_a \mid a \in \mathcal{A}\}$ where $\Sigma^\phi_a(w) = \Sigma_a(w) \cap [\phi]_M$

We add associated public announcement operators $[\phi]$ to the language, interpreted as follows:

$$M, s \models [\phi]\psi \iff M^\phi, s \cap |\phi|_M \models \psi$$

Example

Assume $\Sigma_a(w)$ is constant throughout the model.



Conservative Extension

As with propositional inquisitive logic and classical propositional logic, (dynamic) inquisitive epistemic logic is a conservative extension of (dynamic) epistemic logic.

Given an inquisitive epistemic model $M = (W, V, \Sigma_{\mathcal{A}})$ define an epistemic model $M^e = (W, V, \sigma_{\mathcal{A}})$ by replacing each Σ_a with σ_a, where $\sigma_a(w) := \bigcup \Sigma_a(w)$. Then for formulas α of standard (dynamic) epistemic logic:

$$M, w \models \alpha \iff M^e, w \models \alpha$$

Factive Announcements

We say that ϕ is factive if it does not contain K_a, K_*, E_a, E_*.

As in standard dynamic epistemic logic, the public announcement of a non-factive sentence ϕ need not result in ϕ being publicly entertained ($p \wedge \neg K_* \phi$ still works as an example of this).

However, one can show:

- For factive ϕ, ϕ is publicly entertained after an announcement of ϕ. I.e., $M^{\phi}, s \models E_* \phi$ for all s. I.e., the following is valid: $[\phi] E_* \phi$.
- For factive statements α, a public announcement of α results in common knowledge of α. I.e., $M^{\alpha}, s \models K_* \alpha$ for all s. I.e., the following is valid: $[\alpha] K_* \alpha$.

Factive Question Announcements

For factive questions ϕ, one might expect an announcement of ϕ to lead to ϕ becoming an open public issue, i.e., that the following is valid: $[\phi]W_*\phi$.

But while $[\phi]E_*\phi$ is valid, it might be that ϕ is publicly settled following the announcement, so that $W_*\phi$ is false:

- It might be that ϕ is already publicly settled before the announcement.
- It might be that the announcement of ϕ itself publicly settles ϕ.
  E.g. if $M, w \models q \wedge \neg K_* q \wedge K_* \neg p$, then for $\phi := (p \wedge \neg q) \vee\!\!\!\vee (\neg p \wedge q)$, we have $M, w \models \neg K_*\phi \wedge [\phi]K_*\phi$. The announcement of ϕ creates public knowledge that one of p and q; combined with the existing public knowledge that $\neg p$, there is public knowledge of $\neg p \wedge q$, which is sufficient to resolve ϕ.

However, it can be shown that if common knowledge is sufficient to establish the presupposition α_ϕ of a factive question ϕ but not sufficient to resolve ϕ, then announcing ϕ will lead to ϕ becoming a public open issue. That is, for factive ϕ, the following is valid:

$$(K_*\alpha_\phi \wedge \neg K_*\phi) \rightarrow [\phi]W_*\phi$$

Informativeness and Inquisitiveness Again

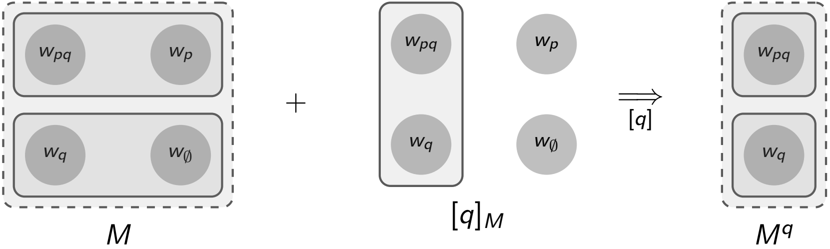
Dynamic epistemic inquisitive semantics allows for publicly sensitive notions of informativeness and inquisitiveness:

- ϕ is *informative* in a world w in case an announcement of ϕ in w enhances the public information state at w: $\sigma_*^\phi(w) \subset \sigma_*(w)$. Equivalently: it is not already common knowledge at w that ϕ is true: $\sigma_*(w) \not\subseteq |\phi|_M$.
- ϕ is *inquisitive* in a world w in case the public issues after an announcement of ϕ are not just the restrictions of the old public issues to the new public information state: $\Sigma_*^\phi(w) \subset \Sigma_*(w) \restriction_{\sigma_*^\phi(w)}$, where for an issue I and state s, $I \restriction_s = \{t \in I \mid t \subseteq s\}$.

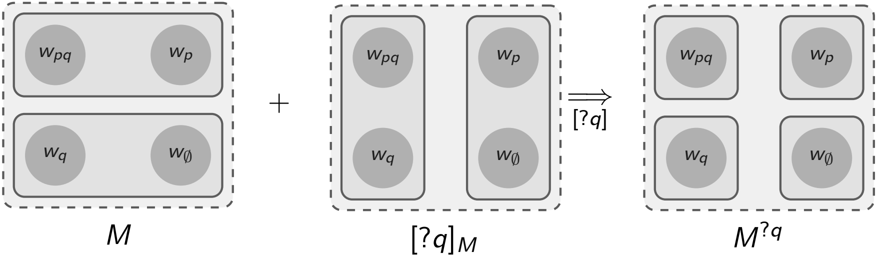
It can be show that statements are never inquisitive in a world.

Examples

Informative, not inquisitive:

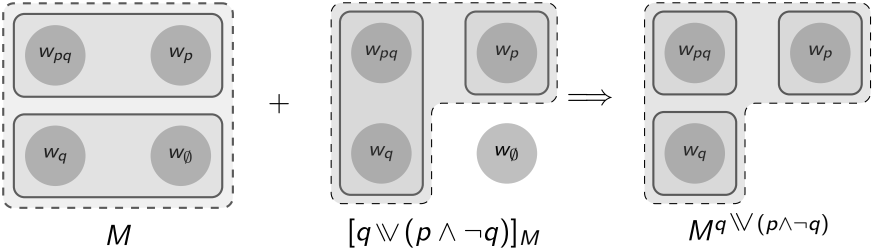


Inquisitive, not informative:



Examples

Inquisitive and informative:



Appropriateness and Division of Communicative Labor

In natural languages, the communicative tasks of providing information and raising issues seem to be rigidly divided between statements and questions.

This can be modelled in the framework of inquisitive dynamic epistemic logic by the following principle of division of communicative labor:

- The announcement of a statement α is *appropriate* in w just in case α is informative but not inquisitive in w.
- The announcement of a question ϕ is *appropriate* in w just in case ϕ is inquisitive but not informative in w.

It can then be shown:

- Announcing a statement α is appropriate in w just in case α is not already common knowledge in w: $M, w \models \neg K_* \alpha$.
- Announcing a question ϕ is appropriate in w just in case the presupposition α_ϕ of ϕ is common knowledge in w and ϕ is not already publicly entertained in w: $M, w \models K_* \alpha_\phi \wedge \neg E_* \phi$.

Pragmatically Sensitive Announcement Operators

One can extend the language with 'pragmatically sensitive' operators $[\phi]^p$ which take the appropriateness conditions into account: $[\phi]^p\psi$ is true in w just in case either ϕ is not appropriate or true in w, or, following a truthful and appropriate announcement of ϕ, ψ is true.

Writing $\lfloor\phi\rfloor_M$ for the set of worlds in M in which an announcement of ϕ is appropriate, the support conditions for $[\phi]^p$ are as follows:

$$M, s \models [\phi]^p\psi \iff M, s \cap \lfloor\phi\rfloor_M \models [\phi]\psi$$

Recall that for factive questions ϕ:

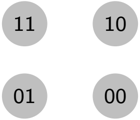
$$(K_*\alpha_\phi \wedge \neg K_*\phi) \rightarrow [\phi]W_*\phi$$

Given that announcing a question ϕ is only appropriate if α_ϕ is common knowledge and ϕ is not publicly entertained (and hence not common knowledge), an appropriate announcement of ϕ always satisfies the antecedent of the above, and hence will result in ϕ becoming a public issue. That is, for factive questions ϕ, the following is valid:

$$[\phi]^p W_*\phi$$

Exercise 1: support conditions

Consider the following possible worlds, each corresponding to a two-digit code in which each digit is either 1 or 0:



For each of the following sentences, draw a picture of the maximal information state in which it is supported:

1. The code is 11.
2. The second digit is 1.
3. If the first digit is 1, the second digit is 1.
4. Is the code 11?
5. What is the first digit?
6. What is the second digit?
7. What is the code?
8. If the first digit is 1, what is the second?

For solution, see P.28 of Ciardelli, Groenendijk & Roelofsen 2018

Exercise 2: inquisitive dynamic epistemic logic

Consider the following inquisitive epistemic model depicting the outcomes of a die roll (the proposition 1 is true only in world w_1, etc.), and a public state state $\Sigma(w)$, assumed to be constant throughout the model:

Let $\alpha_{\text{even}} := 2 \vee 4 \vee 6$; $\alpha_{\text{odd}} := 1 \vee 3 \vee 5$; $\alpha_{\text{low}} := 1 \vee 2$; $\alpha_{\text{med}} := 3 \vee 4$; $\alpha_{\text{hi}} := 5 \vee 6$; $\mu_{\text{parity}} := \alpha_{\text{even}} \mathbin{\vee\!\!\!\vee} \alpha_{\text{odd}}$; $\mu_{\text{range}} := \alpha_{\text{low}} \mathbin{\vee\!\!\!\vee} \alpha_{\text{med}} \mathbin{\vee\!\!\!\vee} \alpha_{\text{hi}}$; $\mu_{\text{out}} := 1 \mathbin{\vee\!\!\!\vee} 2 \mathbin{\vee\!\!\!\vee} 3 \mathbin{\vee\!\!\!\vee} 4 \mathbin{\vee\!\!\!\vee} 5 \mathbin{\vee\!\!\!\vee} 6$.

Determine whether (drawing updated models as required):

- $M, w_1 \models K_* \mu_{\text{even}}$
- $M, w_1 \models W_* \mu_{\text{out}}$
- $M, w_1 \models W_* \mu_{\text{even}}$
- $M, w_1 \models [\mu_{\text{low}}] K_* \mu_{\text{even}}$
- $M, w_1 \models [\mu_{\text{range}}] W_* \mu_{\text{out}}$
- α_{odd} is informative/inquisitive in w_1
- $\alpha_{\text{odd}} \to \alpha_{\text{low}}$ is informative/inquisitive in w_1
- μ_{range} is informative/inquisitive in w_1
- $\alpha_{\text{odd}} \to \mu_{\text{range}}$ is informative/inquisitive in w_1
- $M, w_1 \models [\mu_{\text{range}}]^p W_* \mu_{\text{range}} \to \mu_{\text{parity}}$

Exercise 3: Logical principles in inquisitive epistemic logic

For $\Box \in \{K_a, E_a, W_a\}$, determine whether the following principles are valid. Provide proofs or counterexacmples.

- Normality: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- Distribution laws:
  - $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$
  - $\Box(\phi \vee \psi) \leftrightarrow \Box\phi \vee \Box\psi$
- Monotonicity: if $\phi \models \psi$, then $\Box\phi \models \Box\psi$
- Necessitation: if $\models \phi$, then $\models \Box\phi$