Compositionality, Negation and Modality

Aleksi Anttila Marco Degano

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Outline

1. Compositionality
2. Negation
3. Modality

Semantic type

Let syntax be generated by operations c, and let μ assign meanings.

- Two ways to think about compositionality.
- Functional: For each n-ary c, some operation F_c satisfies

$$\mu(c(\vec{\alpha})) = F_c(\mu(\vec{\alpha}))$$

- Substitutional: If $\mu(\alpha_i) = \mu(\beta_i)$, then every meaningful context $\pi[-]$ preserves the equality:

$$\mu(\pi[\alpha_i]) = \mu(\pi[\beta_i])$$

- For total semantics, or when the interpreted domain is closed under subterms, the two formulations coincide.
- The hard question is: which values make them true without trivializing meaning?

Hodges 1997, 1998; Westerståhl 1998; Pagin and Westerståhl 2010; Baltag, van Benthem, and

Contextuality

Suppose μ initially interprets only expressions in X. For example, only sentences.

An expression $e \notin X$ may still occur inside interpreted expressions:
$e \quad \rightsquigarrow \quad \pi[e] \in X$.

Two expressions have the same Fregean value when they fit into the same interpreted contexts and no such context distinguishes them:

$$e \equiv_\mu^F f \quad \Longleftrightarrow \quad e \sim_\mu f \ \text{ and } \ \forall \pi[-]\,(\pi[e] \in X \Rightarrow \mu(\pi[e]) = \mu(\pi[f]))$$

The Fregean value of e is its complete observable behavior:

$$|e|_\mu^F = \{f : f \equiv_\mu^F e\}.$$

Hodges 2001, 2011

Hodges's Lifting Lemma

If every expression occurs inside some expression in X, then $e \mapsto |e|^F_\mu$ is a total, compositional semantics.

$$|e|^F_\mu = |f|^F_\mu \quad \Longleftrightarrow \quad \text{no interpreted context distinguishes } e \text{ and } f.$$

Hodges 2001, 2011

IF logic

In independence-friendly logic, $(\exists y/\{x\})\varphi$ requires the choice of y to be independent of the value of x.

- At a subformula, legal strategies depend on the information sets inherited from the larger game.
- An open subformula therefore need not determine an ordinary subgame.
- Its pointwise extension records which assignments work separately, but not whether *one uniform strategy* works across them.

The challenge is to find denotations rich enough to recover inherited strategic information.

Hintikka and Sandu 1989; Hintikka 1996

Singleton truth and uniformity

Let M have two distinct elements a, b, and consider $\psi(x) := (\exists y/\{x\})\, x = y$.

	x
i_0	a
i_1	b

- On each singleton team, Eloise can choose $y = i(x)$.
- On $X = \{i_0, i_1\}$, she must choose y uniformly, independently of x.
- No single choice works for both a and b.

- $M, \{i_0\} \vDash \psi$
- $M, \{i_1\} \vDash \psi$
- But $M, \{i_0, i_1\} \nvDash \psi$
- As we have seen, truth on all points does not determine truth on the team.

Universal lifting

Suppose open formulas had only ordinary Tarskian extensions

$$[\![\varphi]\!]^M \subseteq M^{\mathrm{FV}(\varphi)}$$

and team support were obtained by universal lifting:

$$M, X \vDash^+ \varphi \quad \Longleftrightarrow \quad X \subseteq [\![\varphi]\!]^M$$

Then:

$$M, X \vDash^+ \varphi \quad \Longleftrightarrow \quad \forall i \in X \; M, \{i\} \vDash^+ \varphi$$

Every formula would be *flat*. But $\psi(x) = (\exists y/\{x\})x = y$ is supported by both singleton subteams of $\{i_0, i_1\}$ and not by their union.

Hodges 1997; Väänänen 2007

Arbitrary sets of assignments

The flatness argument rules out one lifting recipe. Cameron and Hodges prove a stronger result. There is no *total compositional* semantics for all IF formulas that

1. agrees with game-theoretic truth on IF sentences, and
2. assigns every open formula merely a set of assignments.

For every $n \geq 2$, some n-element model requires more than 2^n distinct values for formulas $\varphi(x)$, while subsets of $M^{\{x\}}$ supply only 2^n candidates.

Shifting the parameter

$t_M(\varphi, X) = 1$ iff $M, X \vDash \varphi$.

For split disjunction, $t_M(\varphi \vee \psi, X) = 1 \iff \exists Y, Z(X = Y \cup Z \;\&\; t_M(\varphi, Y) = 1 \;\&\; t_M(\psi, Z) = 1)$

The parameter X is shifted: the constituents are evaluated at Y and Z, not at the original X. Hence truth-at-X is not, by itself, a compositional value.

Instead of assigning φ a truth value at one team, we assign it its full truth profile: the function mapping each team to 1 or 0, equivalently the set of all teams that support it.

$$\widehat{t}_M(\varphi) := \lambda X.\, t_M(\varphi, X) \quad \cong \quad \{X \mid M, X \vDash \varphi\}$$

Baltag, van Benthem, and Westerståhl 2023, Pagin 2005

Teams

For a variable domain V, let $\mathsf{Teams}_M(V) = \mathcal{P}(M^V)$. Hodges's positive denotation is the collection of the supporting teams:

$$[\![\varphi]\!]^+_M = \{X \in \mathsf{Teams}_M(\mathrm{FV}(\varphi)) \mid M, X \vDash^+ \varphi\}$$

Algebra on team properties

Fix V. Let $P, Q \subseteq \mathrm{Teams}_M(V)$. For $E_{y/W}$ let $P \subseteq \mathrm{Teams}_M(V \cup \{y\})$.

$$P \cap Q = \{X \mid X \in P \text{ and } X \in Q\},$$
$$P \oplus Q = \{X \mid \exists Y \in P \, \exists Z \in Q \; (X = Y \cup Z)\},$$
$$E_{y/W}(P) = \{X \mid \exists \; W\text{-uniform } F : X \to M \; X[F/y] \in P\}$$

Here F is W-uniform iff

$$i \restriction (\mathrm{Dom}(X) \setminus W) = j \restriction (\mathrm{Dom}(X) \setminus W) \quad \Rightarrow \quad F(i) = F(j)$$

Then the positive clauses become genuine operations on denotations:

$$\llbracket \varphi \wedge \psi \rrbracket^+_M = \llbracket \varphi \rrbracket^+_M \cap \llbracket \psi \rrbracket^+_M$$
$$\llbracket \varphi \vee \psi \rrbracket^+_M = \llbracket \varphi \rrbracket^+_M \oplus \llbracket \psi \rrbracket^+_M$$
$$\llbracket (\exists y/W)\varphi \rrbracket^+_M = E_{y/W}(\llbracket \varphi \rrbracket^+_M)$$

Hodges 1997; Väänänen 2007

Dual negation forces bilateral meanings

A positive family alone does not in general determine the positive family of the dual negation

$$\mu_M(\varphi) = \langle P_\varphi, N_\varphi \rangle = \langle \{X \mid M, X \vDash^+ \varphi\}, \{X \mid M, X \vDash^- \varphi\} \rangle$$

Hodges 1997, Anttila 2026

Outline

Some common negation operators in team semantics

- Boolean negation $\sim$

$$s \vDash {\sim}\phi \iff s \nvDash \phi$$

- Bilateral/dual/game-theoretical negation

$$s \vDash \neg\phi \iff s \Dashv \phi$$

- Intuitionistic negation

$$s \vDash \neg_i\phi \iff t \nvDash \phi \text{ for all nonempty } t \subseteq s$$

- 'Classical' negation $\neg_c$: only allow negation to occur in front of formulas in the classical fragment of the language (or only in front of propositional variables):

$$s \vDash \neg_c\alpha \iff \{v\} \nvDash \alpha \text{ for all } v \in s$$

Boolean negation ∼

$$s \vDash {\sim}\phi \iff s \nvDash \phi$$

- Unlike with most other team-semantic negations, bivalence holds:

$$t \vDash \phi \text{ or } t \vDash {\sim}\phi$$

- Most team-based logics are designed to conservatively extend some singleton-evaluated logic, with the negation $\neg_T$ of the team logic extending the negation $\neg_S$ of the singleton logic via flatness/singleton-correspondence: for all ϕ of the singleton logic

$$t \vDash \neg_T\phi \iff \forall v \in t : \{v\} \vDash \neg_T\phi \iff v \vDash \neg_S\phi$$

  ∼ does not extend a non-team negation in this way.
- Conjunction and negation form a functionally complete set of connectives. With ∼ and $\wedge$, we therefore have classical propositional logic over teams.
- Addition of ∼ to a logic will typically break all closure properties and lead to a great increase in expressive power. E.g., classical propositional logic extended with ∼ can express all sets of teams; dependence logic with ∼ is expressively equivalent with full second-order logic.

Väänänen 2007, Kontinen & Nurmi 2011, Kontinen et al. 2015

Bilateral/game-theoretic/dual negation

$$s \vDash \neg\phi \iff s \dashv \phi$$

- Specified with two semantic relations (support/rejection; existence of winning strategy for verifier/falsifier in semantic game).
- Support typically interpreted as assertability in an information state; rejection as rejectability.
- Bilateral meanings: positive meanings $\|\phi\|^+ = \{t \mid t \vDash \phi\}$ do not (necessarily) determine negative $\|\phi\|^- = \{t \mid t \vDash \neg\phi\}$.
- How should $\|\phi\|^+$ and $\|\phi\|^-$ relate to one another?

Intuitionistic negation $\neg_i$

$$s \vDash \neg_i \phi \iff \forall t \subseteq s: \text{ if } t \vDash \phi, \text{ then } t = \varnothing$$

- Definable using the intuitionistic implication $\rightarrow$: $\neg_i := \phi \rightarrow \bot$
- Forces downward closure.
- In the information-state interpretation, support of $\neg_i \phi$ in an information state can be interpreted as the information expressed by ϕ being incompatible with the state: if the state is enhanced with this information (take any $t \subseteq s$ with $t \vDash \phi$), the resulting state is the state of absurdity ($t = \varnothing$).

Väänänen 2007, Yang 2013

Dual negation: extreme failure of determination

With the dual negation preservation of equivalents under replacement does not hold in the scope of negation:

$$\neg\forall x(\exists y/\{x\})x = y \equiv \bot \quad \text{but} \quad \neg\neg\forall x(\exists y/\{x\})x = y \equiv \forall x(\exists y/\{x\})x = y \not\equiv \top \equiv \neg\bot$$
$$\neg\mathsf{ne} \equiv \bot \quad \text{but} \quad \neg\neg\mathsf{ne} \equiv \mathsf{ne} \not\equiv \top \equiv \neg\bot.$$

Put another way, the positive denotation $\|\phi\|^+$ of ϕ does not determine the negative $\|\phi\|^-$:

$$\|\neg\mathsf{ne}\|^+ = \|\bot\|^+ \qquad \text{but} \qquad \|\neg\mathsf{ne}\|^- \neq \|\bot\|^-.$$

Burgess showed that for IF sentences, this failure of determination is extreme: for an IF sentence, given only $\|\phi\|^+ = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$, $\|\phi\|^- = \{\mathcal{M} \mid \mathcal{M} \vDash \neg\phi\}$ can be any class of models disjoint from $\|\phi\|^+$ that is expressible in IF.

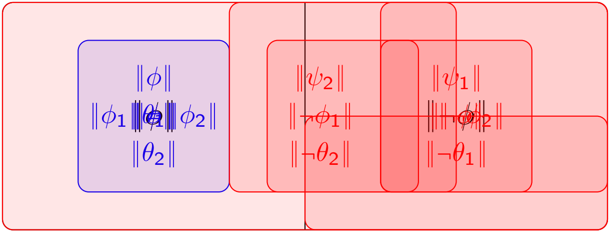
Burgess 2003

Burgess' Theorem

In classical first-order logic, $\mathcal{M} \vDash \neg\phi \iff \mathcal{M} \nvDash \phi$, so given $\|\phi\| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$, to find $\|\neg\phi\|$, simply take the complement of $\|\phi\|$.

In a setting in which $\mathcal{M} \vDash \neg\phi \not\Longleftrightarrow \mathcal{M} \nvDash \phi$, we can ask how $\|\phi\|$ and $\|\neg\phi\|$ ought to relate to one another.

Burgess: for any IF-sentences ϕ and ψ, if $\|\phi\|$ and $\|\psi\|$ are disjoint, there is an IF-sentence θ with $\|\theta\| = \|\phi\|$ and $\|\neg\theta\| = \|\psi\|$.



So if we only know $\|\phi\|$, $\|\neg\phi\|$ can be any class of models $\mathcal{P}$, as long as that class is IF-definable ($\mathcal{P} = \|\psi\|$ for some ψ) and disjoint with $\|\phi\|$.

Assume all models are of size ≥ 2.

Lemma: There is θ_0 s.t. $\theta_0 \equiv \perp \equiv \neg\theta_0$. (E.g. the matching pennies sentence $\forall x(\exists y/\{x\})x = y$.)

Separation Theorem: If $\|\phi\|$ and $\|\psi\|$ are disjoint, there is an η s.t. $\phi \vDash \eta$ and $\psi \vDash \neg\eta$. (Follows from ESO-equivalence and Craig's interpolation for classical first-order logic.)

Burgess' Theorem for IF: For any IF-sentences ϕ, ψ, 1 implies 2:

- 1. $\|\phi\|$ and $\|\psi\|$ are disjoint (i.e., $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$; i.e., $\phi, \psi \vDash \bot$).
- 2. There is an IF-sentence θ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$ (i.e., $\|\phi\| = \|\theta\|$ and $\|\psi\| = \|\neg\theta\|$).

Let $\phi_0 := \phi \vee \theta_0$ and $\psi_0 := \psi \vee \theta_0$ with θ_0 from Lemma. Then:

$$\begin{array}{lllllllll} \phi_0 & \equiv & \phi \vee \theta_0 & \equiv & \phi \vee \bot & \equiv & \phi & & \\ \neg\phi_0 & \equiv & \neg(\phi \vee \theta_0) & \equiv & \neg\phi \wedge \neg\theta_0 & \equiv & \neg\phi \wedge \bot & \equiv & \bot \end{array}$$

Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \bot$. By Separation let η be s.t. $\phi_0 \vDash \eta$ and $\psi_0 \vDash \neg\eta$. Let $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$. Then:

$$\begin{array}{llll} \theta & \equiv \phi_0 \wedge (\neg\psi_0 \vee \eta) & \equiv \phi_0 \wedge (\bot \vee \eta) & \equiv \phi_0 \wedge \eta \\ & & \equiv \phi_0 & \equiv \phi \\ \neg\theta & \equiv \neg(\phi_0 \wedge (\neg\psi_0 \vee \eta)) & \equiv \neg\phi_0 \vee \neg(\neg\psi_0 \vee \eta) & \equiv \bot \vee (\neg\neg\psi_0 \wedge \neg\eta) \\ & \equiv \psi_0 \wedge \neg\eta & \equiv \psi_0 & \equiv \psi \end{array}$$

Burgess' intended his theorem to serve in part as a point against IF and Hintikka's philosophical ambitions:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety [dual negation], the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

With bilateral meanings $(\|\phi\|^+, \|\phi\|^-)$, and assuming double negation elimination, the negation corresponds simply to the operation flipping the positive and negative denotations:
$\neg : (\|\phi\|^+, \|\phi\|^-) \mapsto (\|\phi\|^-, \|\phi\|^+)$.

Burgess 2003, Hodges 1997, Kontinen & Väänänen 2011

Expressive completeness

A logic L is expressively complete for some class $\mathbb{P}$ if

$$\mathbb{P} = \|L\|, \qquad \text{where } \|L\| = \{\|\phi\| \mid \phi \in L\}.$$

We use $\|\phi\|$ to refer to the set of semantic objects satisfying ϕ:

- In propositional team semantics, $\{t \mid t \vDash \phi\}$.
- In modal (Kripke model) team semantics, $\{(M, t) \mid M, t \vDash \phi\}$.
- In first-order team semantics, $\{(\mathcal{M}, X) \mid \mathcal{M}, X \vDash \phi\}$ (for sentences ϕ, this can be simplified to $\{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$).

Some expressive completeness results

- Over sentences, dependence logic and IF logic are both expressively complete for the class of ESO-definable sets of FO structures.
- Over formulas, dependence logic and IF logic are both expressively complete for the class of ESO-definable downward-closed and empty-team closed sets of pointed FO structures.
- Propositional dependence logic and propositional inquisitive logic are both expressively complete for the class of downward-closed and empty-team-closed sets of propositional teams.
- BSML is expressively complete for the class of convex, union-closed, and bisimulation-invariant sets of pointed Kripke models.

Intuitively, expressive completeness of e.g. propositional dependence logic says that not only are all meanings of dependence logic downward- and empty-team-closed, but propositional dependence logic can capture all such meanings.

Väänänen 2007, Kontinen & Väänänen 2009, Ciardelli & Roelofsen 2011, Yang & Väänänen 2016, Anttila & Knudstorp 2025

Bicompleteness

Burgess' theorem can be viewed as an expressive completeness result with respect to bilateral meanings.

Let $\|\phi\|^{\pm}$ denote the pair

$$(\|\phi\|, \|\neg\phi\|),$$

and for a logic L,

$$\|L\|^{\pm} = \{\|\phi\|^{\pm} \mid \phi \in L\}.$$

A logic is bicomplete for a class of pairs $\mathbb{P}$ if $\|L\|^{\pm} = \mathbb{P} \cap (\|L\| \times \|L\|)$.

Burgess 2003, Dechesne 2005, Anttila 2026

Bicompleteness for IF sentences

Burgess' theorem, together with its converse (which implies: $\mathcal{M} \vDash \phi$ and $\mathcal{M} \vDash \neg\phi$ are never true simultaneously—$\|\phi\|^+ \cap \|\phi\|^- = \varnothing$) shows that with respect to sentences, IF is bicomplete for disjoint pairs.

The positive expressive completeness theorems for IF say that IF can express all and only meanings $\|\phi\|$ that are ESO-expressible, downward- and empty-team closed.

The bicompleteness theorem says that among the IF-expressible pairs, the ones corresponding to a bilateral meaning $\|\phi\|^\pm$ are precisely the disjoint ones. In other words, the property of pairs captured by the IF dual negation is disjointness.

Bicompleteness for IF formulas

Say that formulas ϕ and ψ are $\bot$-incompatible if $X \vDash \phi$ and $X \vDash \psi$ together imply $X = \varnothing$. I.e., $\phi, \psi \vDash \bot$.

Similarly, a class of pairs $(\mathcal{P}, \mathcal{Q})$, (where $\mathcal{P}$, $\mathcal{Q}$ are sets of teams/pointed models) is $\bot$-incompatible if whenever $X \in \mathcal{P} \cap \mathcal{Q}$, $X = \varnothing$.

With respect to formulas, IF is bicomplete for $\bot$-incompatible pars.

The property of pairs captured is $\bot$-incompatibility.

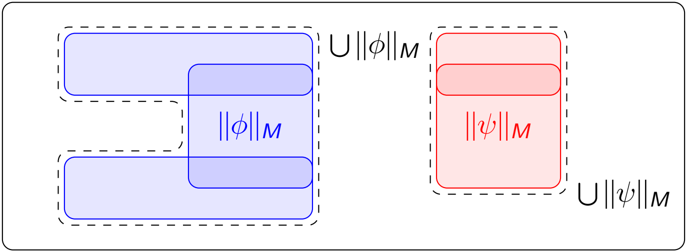
Kontinen & Väänänen 2011, Anttila 2026

Bicompleteness for BSML

Let the ground team $|\phi|_M$ of ϕ in a model M be $\bigcup \|\phi\|^+_M$.

Say that formulas ϕ and ψ are ground-incompatible if $M, s \vDash \phi$ and $M, t \vDash \psi$ together imply $s \cap t = \varnothing$. Equivalently, $|\phi|_M \cap |\psi|_M = \varnothing$ for all M.

BSML is bicomplete for ground-incompatible pairs.



Ground-incompatibility and $\bot$-incompatibility

Ground-incompatibility implies $\bot$-incompatibility: for ground-incompatible ϕ and ψ, if $X \vDash \phi \wedge \psi$, then $X = X \cap X = \varnothing$.

In a downward-closed setting, the two are equivalent: for $\bot$-incompatible downward-closed ϕ and ψ, if $X \vDash \phi$ and $Y \vDash \psi$, then $X \cap Y \vDash \phi \wedge \psi$, so $X \cap Y = \varnothing$.

Ground-incompatibility in information states

Interpret teams as information states and extend classical logic with the epistemic 'might'-operator:

$$X \vDash \blacklozenge\phi \iff \exists Y \subseteq X : Y \vDash \phi \text{ and } Y \neq \varnothing$$

$\blacklozenge r$ *It might be raining.*

The failure of downward closure can be taken to represent epistemic uncertainty, and the ground-team of a formula ϕ the factual, non-epistemic information expressed by ϕ. E.g., $|q \wedge \blacklozenge r|_M = |q|_M$.

Then ground-incompatibilities $|r|_M \cap |\neg r|_M = \varnothing$ represent contradictions in factual information.

If ϕ and ψ are $\bot$-incompatible but not ground-incompatible, it is not only factual information but also information regarding epistemic uncertainty that makes them incompatible. E.g., epistemic contradictions:

"It is not raining but it might be raining"

$$\neg r, \blacklozenge r \vDash \bot \qquad |\neg r|_M \cap |\blacklozenge r|_M = |\neg r|_M \neq \varnothing.$$

More incompatibility notions/pair properties

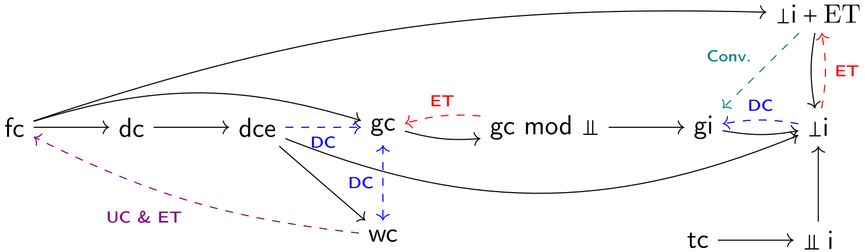
There are many other incompatibility notions captured by different team logics with dual negations.

Say that ϕ and ψ are ground-complementary if $|\phi|_M = |\top|_M \setminus |\psi|_M$.

Propositional dependence logic with the dual negation is bicomplete for ground-complementary pairs.

Pair property	Definition(s)	`Bicomplete` logics
$\perp$-incompatible ($\perp$i)	• $[s \vDash \phi_{\mathbf{0}}$ and $s \vDash \phi_{\mathbf{1}}] \implies s = \varnothing$ • $\phi_{\mathbf{0}}, \phi_{\mathbf{1}} \vDash \perp$ • $\perp$i + ET or $\perp\!\!\!\perp$ i	D, IF
Ground-incompatible (gi)	• $[s \vDash \phi_{\mathbf{0}}$ and $t \vDash \phi_{\mathbf{1}}] \implies s \cap t = \varnothing$ • $\lvert\phi_{\mathbf{0}}\rvert \cap \lvert\phi_{\mathbf{1}}\rvert = \varnothing$	D, IF, BSML, BSML($\mathbin{\mathpalette\vee\vee}$), PL(ne, $\mathbin{\mathpalette\vee\vee}$)
$\perp\!\!\!\perp$-incompatible ($\perp\!\!\!\perp$ i)	• $\phi_{\mathbf{0}}$ and $\phi_{\mathbf{1}}$ never jointly true • $\phi_{\mathbf{0}}, \phi_{\mathbf{1}} \vDash \perp\!\!\!\perp$ • $\lVert\phi_{\mathbf{0}}\rVert \cap \lVert\phi_{\mathbf{1}}\rVert = \varnothing = \lVert\perp\!\!\!\perp\rVert$	
$\perp$-incompatible and empty team prop. ($\perp$i + NE)	• $[s \vDash \phi_{\mathbf{0}}$ and $s \vDash \phi_{\mathbf{1}}] \iff s = \varnothing$ • $\phi_{\mathbf{0}}, \phi_{\mathbf{1}} \vDash \perp$ and $\phi_{\mathbf{0}}, \phi_{\mathbf{1}} \nvDash \perp\!\!\!\perp$ • $\lVert\phi_{\mathbf{0}}\rVert \cap \lVert\phi_{\mathbf{1}}\rVert = \{\varnothing\} = \lVert\perp\rVert$	D, IF
World-complementary (wc)	• $\{w\} \vDash \phi_{\mathbf{0}} \iff \{w\} \nvDash \phi_{\mathbf{1}}$	PL(=(·)), PL
Team-complementary (tc)	• $s \vDash \phi_i \iff s \nvDash \phi_{\mathbf{1}-i}$ • $\lVert\phi_i\rVert = \lVert\top\rVert \setminus \lVert\phi_{\mathbf{1}-i}\rVert$	PL($\sim$) (w.r.t. $\sim$)
Flat-complementary (fc)	• wc and $\phi_{\mathbf{0}}, \phi_{\mathbf{1}}$ flat • $\lVert\phi_i\rVert = \wp(\lvert\top\rvert \setminus \lvert\phi_{\mathbf{1}-i}\rvert)$ • $\lVert\phi_i\rVert = \{s \mid t \vDash \phi_{\mathbf{1}-i} \implies s \cap t = \varnothing\}$ • $\lVert\phi_i\rVert = \bigcup\{P \subseteq \lVert\top\rVert \mid P, \lVert\phi_{\mathbf{1}-i}\rVert$ G-I$\}$	PL
$\phi_{\mathbf{1}}$ down-set complement (dc) of $\phi_{\mathbf{0}}$	• $s \vDash \phi_{\mathbf{1}} \iff$ $[[t \vDash \phi_{\mathbf{0}}$ and $t \subseteq s] \implies t = \varnothing]$	InqB (w.r.t. $\neg_i$), PL
Down-set complements (on either side) (dce)	• $\phi_{\mathbf{1}}$ dc of $\phi_{\mathbf{0}}$ or $\phi_{\mathbf{0}}$ dc of $\phi_{\mathbf{1}}$	HS, PL
Ground-complementary (gc)	• $\lvert\phi_i\rvert = \lvert\top\rvert \setminus \lvert\phi_{\mathbf{1}-i}\rvert$	PL(=(·)), PL
Ground-complementary mod $\perp\!\!\!\perp$	• $\lvert\phi_i\rvert = \lvert\top\rvert \setminus \lvert\phi_{\mathbf{1}-i}\rvert$ or $\phi_{\mathbf{0}} \equiv \perp\!\!\!\perp$ or $\phi_{\mathbf{1}} \equiv \perp\!\!\!\perp$	PL(ne), PL(=(·)), PL
All pairs		PL(ne*, $\mathbin{\mathpalette\vee\vee}$)

Relations between incompatibility notions



Bicompleteness and indetermination

We can think of bicompleteness theorems with respect to different incompatibility notions as measuring the extent to which the positive and negative meanings are independent, with a stronger notion corresponding to less independence:

- Classical logic interpreted on teams is bicomplete for pairs $(\mathcal{P}, \mathcal{Q})$ where $\mathcal{P}$ and $\mathcal{Q}$ are both flat, and $\mathcal{P} = \|\top\| \setminus \mathcal{Q}$. $\|\phi\|$ completely determines $\|\neg\phi\|$, and vice versa.
- Propositional dependence logic is bicomplete for ground-complementary pairs. $\|\phi\|$ does not determine $\|\neg\phi\|$, but does determine $|\neg\phi|$.
- BSML is bicomplete for ground-complementary pairs. $\|\phi\|$ does not determine $\|\neg\phi\|$ or $|\neg\phi|$, but we know $|\phi|$ and $|\neg\phi|$ are disjoint.
- If a logic is bicomplete for $\perp$-incompatible pairs but not for ground-complementary pairs, we do not even know that, but we know $\|\phi\| \cap \|\neg\phi\| \subseteq \{\varnothing\}$.

Outline

1. Compositionality

2. Negation

3. Modality

Modality over information states

In ordinary Kripke semantics, formulas denote sets of worlds $[\![\varphi]\!]_M \subseteq W$ and a modal operator transforms one proposition into another:

$$\mathcal{P}(W) \longrightarrow \mathcal{P}(W)$$

In team semantics, formulas are evaluated at information states $T \subseteq W$. Their denotations are therefore sets of states:

$$[\![\varphi]\!]_M = \{T \subseteq W : M, T \vDash \varphi\} \subseteq \mathcal{P}(W)$$

Hence a modal operator acts at the higher semantic type

$$\mathcal{P}(\mathcal{P}(W)) \longrightarrow \mathcal{P}(\mathcal{P}(W))$$

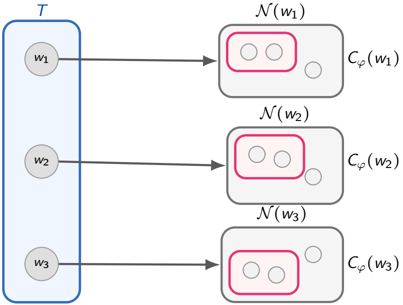
- Starting from an input state T, the modality determines one or more admissible successor states.
- It then specifies how the complement is evaluated over those successors.
- Variety of modalities: different choices yield genuinely different modal operators.

Some independent choices

1. Should modality move to one collective successor state or many local ones?
2. Should the model associate worlds with worlds or with states?
3. Should the modal complement see only pointwise truth, or also relations among possibilities?

Local modalities

- A local modality does not build one joint successor team for the whole input team.
- For each source world $w \in T \subseteq W$, look at successor information states available at w.
- The complement is evaluated locally, at successor states attached to each single w.
- Local modalities have the form: $M, T \vDash O\varphi \iff \forall w \in T\ C_\varphi(w)$



General local-state models

- Let $M = (W, V, \mathcal{N})$, where $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ is the family of admissible successor states at w.
- Usually we require successor states to be nonempty.
- For a Kripke frame (W, R), the most common choice is:

$$\mathcal{N}_R(w) = \mathcal{P}^+(R[w]) = \{s \subseteq R[w] : s \neq \varnothing\}$$

- The pooled successor range is:

$$\rho(w) = \bigcup \mathcal{N}(w)$$

- If $\mathcal{N}(w) = \mathcal{P}^+(R[w])$, then:

$$\rho(w) = R[w]$$

- The local architecture separates two questions:
  - which source worlds $w \in T$ must be checked?
  - which successor states $S \in \mathcal{N}(w)$ must support the complement?

Three basic local modalities

Fix a state formula φ.

- Existential-state modality:

$$M, T \vDash \Diamond_{\mathcal{N}} \varphi \iff \forall w \in T \; \exists S \in \mathcal{N}(w) \; M, S \vDash \varphi$$

At every live source world, some admissible successor state supports φ.

- Universal-state modality:

$$M, T \vDash \boxplus_{\mathcal{N}} \varphi \iff \forall w \in T \; \forall S \in \mathcal{N}(w) \; M, S \vDash \varphi$$

At every source world, every admissible successor state supports φ.

- Pooled-state modality:

$$M, T \vDash \Box_{\cup} \varphi \iff \forall w \in T \; M, \rho(w) \vDash \varphi$$

At every source world, the total pooled successor range supports φ.

The local quantifier square

The source-world quantifier and the successor-state quantifier can be varied independently.

- Universal source, existential state ($\forall\exists$):

$$\forall w \in T\ \exists S \in \mathcal{N}(w)\ S \vDash \varphi$$

- Universal source, universal state ($\forall\forall$):

$$\forall w \in T\ \forall S \in \mathcal{N}(w)\ S \vDash \varphi$$

- Existential source, existential state ($\exists\exists$):

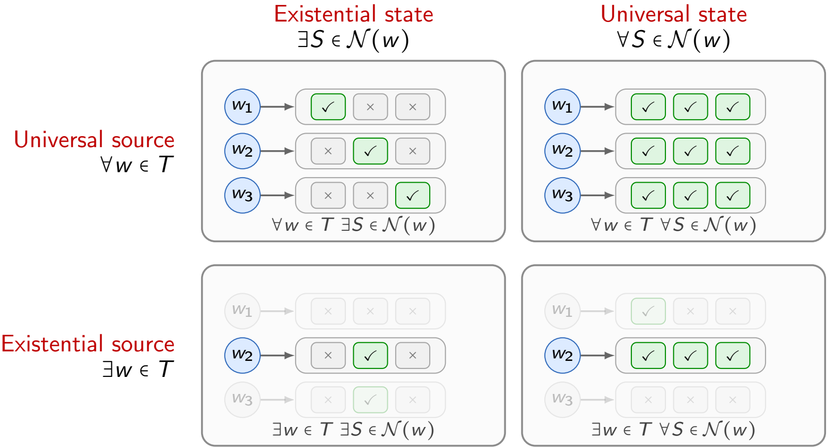
$$\exists w \in T\ \exists S \in \mathcal{N}(w)\ S \vDash \varphi$$

- Existential source, universal state ($\exists\forall$):

$$\exists w \in T\ \forall S \in \mathcal{N}(w)\ S \vDash \varphi$$

- The first two are the standard local modalities.
- The last two say that some live possibility has the relevant modal property, with the latter requiring that this live source possibility is robustly safe.

The local quantifier square



Universal-source modalities are flat at the input

- Let $O \in \{\Diamond_{\mathcal{N}}, \boxplus_{\mathcal{N}}, \Box_{\cup}\}$
- Each of these modalities is lifted universally over the input team:

$$M, T \vDash O\varphi \quad \Longleftrightarrow \quad \forall w \in T\ M, \{w\} \vDash O\varphi$$

- Hence $O\varphi$ is flat with respect to the source team T.
- This remains true even when φ itself is non-flat.
- Local modality therefore has a reset effect:

$$\text{team-sensitive complement} \quad \mapsto \quad \text{flat modal formula at the source level}$$

- They are also monotone in their complement:

$$\text{If } \varphi \vDash \psi, \text{ then } O\varphi \vDash O\psi$$

Universal-state versus pooled-state

$$M, T \vDash \boxplus_{\mathcal{N}}\varphi \iff \forall w \in T\ \forall S \in \mathcal{N}(w)\ S \vDash \varphi$$
$$M, T \vDash \Box_{\cup}\varphi \iff \forall w \in T\ \rho(w) \vDash \varphi$$

Algebraically:

$$\boxplus_{\mathcal{N}}\varphi : \mathcal{N}(w) \subseteq [\![\varphi]\!] \qquad \Box_{\cup}\,\varphi : \bigcup \mathcal{N}(w) \in [\![\varphi]\!]$$

- $\boxplus_{\mathcal{N}}$ tests every admissible alternative separately.
- $\Box_{\cup}$ tests only the union of all alternatives.
- Thus $\boxplus_{\mathcal{N}}$ is alternative-sensitive.
- $\Box_{\cup}$ is range-sensitive: it forgets the internal decomposition of $\mathcal{N}(w)$.
- If φ is union closed, then $\boxplus_{\mathcal{N}}\varphi \vDash \Box_{\cup}\varphi$.
- If φ is downward closed, then $\Box_{\cup}\varphi \vDash \boxplus_{\mathcal{N}}\varphi$.
- BSML uses: $\Diamond_{\mathcal{N}_R}$ and $\Box_{\cup}$.
- Inquisitive modal logic uses: $\Box_{\cup}$ and $\boxplus_{\Sigma}$.

Ordinary modal idempotence

For ordinary propositional meanings $X \subseteq W$:

$$\Diamond_R(X) = \{w : \exists v(wRv \;\&\; v \in X)\} \qquad \Box_R(X) = \{w : R[w] \subseteq X\}$$

Applying a modality twice replaces R by R^2:

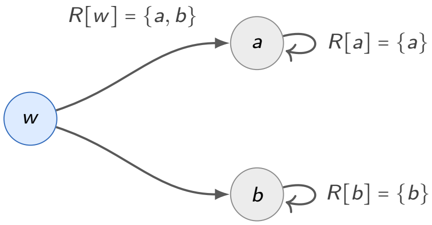
$$\Diamond_R^2(X) = \Diamond_{R^2}(X) \qquad \Box_R^2(X) = \Box_{R^2}(X)$$

Thus, as identities on all propositions X:

$$\Diamond_R^2 = \Diamond_R \iff \Box_R^2 = \Box_R \iff R^2 = R$$

- Ordinary modality only compares the sets of worlds reachable in one and two steps.
- It does not ask how the two-step range is distributed among the intermediate successors.

Example



- One step: $R[w] = \{a, b\}$
- Two steps: $R^2[w] = \{a, b\}$
- The two-step range recovers exactly the one-step range.
- Recovery may be distributed: a preserves only a, while b preserves only b.

Families of states

For local state modalities over $\mathcal{N}_R(w) = \mathcal{P}^+(R[w])$, one step at w tests the entire family $\mathcal{P}^+(R[w])$.

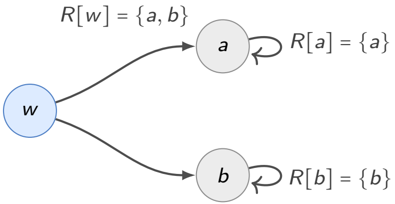
Two local steps test instead $\bigcup_{v \in R[w]} \mathcal{P}^+(R[v])$.

So idempotence on all state properties $A \subseteq \mathcal{P}(W)$ requires:

$$\mathcal{P}^+(R[w]) = \bigcup_{v \in R[w]} \mathcal{P}^+(R[v])$$

- This is stronger than $R^2[w] = R[w]$.
- It requires preservation of the available successor states, not merely preservation of reachable worlds.

The same frame fails locally



- One alternative-sensitive local step at w quantifies over

$$\mathcal{P}^{+}(R[w]) = \{\{a\}, \{b\}, \{a, b\}\}$$

- Two local steps quantify over

$$\mathcal{P}^{+}(R[a]) \cup \mathcal{P}^{+}(R[b]) = \{\{a\}, \{b\}\}$$

- Thus the collective state $\{a, b\}$ is available in one step, but disappears after the local move to a or b.

Idempotence of alternative-sensitive local modalities

The local existential-state modality $\Diamond_{\mathcal{N}_R}$ and the local universal-state modality $\boxplus_{\mathcal{N}_R}$ are idempotent:

$$\Diamond_{\mathcal{N}_R} \Diamond_{\mathcal{N}_R} \varphi \equiv \Diamond_{\mathcal{N}_R} \varphi \text{ and } \boxplus_{\mathcal{N}_R} \boxplus_{\mathcal{N}_R} \varphi \equiv \boxplus_{\mathcal{N}_R} \varphi$$

for every state formula φ exactly when every nonterminal world w satisfies:

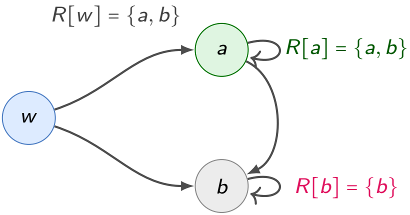
$$\forall v \in R[w]\ R[v] \subseteq R[w] \qquad \text{and} \qquad \exists v_0 \in R[w]\ R[v_0] = R[w]$$

- The first condition is transitivity: a further local step introduces no new possibilities.
- The second condition requires some successor v_0 to carry a copy of the entire successor range of w.
- Ordinary modal idempotence requires only $R^2 = R$ and therefore permits distributed recovery across different successors.
- Alternative-sensitive local idempotence requires one successor to preserve the whole range at once.

Not enough

What about $\Box_\cup \Box_\cup \varphi \equiv \Box_\cup \varphi$?

$$M, T \vDash \Box_\cup \varphi \quad \Longleftrightarrow \quad \forall w \in T\ M, R[w] \vDash \varphi$$



- Suppose $[\![\varphi]\!]_M = \{\{a, b\}\}$. Then $M, \{w\} \vDash \Box_\cup \varphi$
- But $M, \{w\} \nvDash \Box_\cup \Box_\cup \varphi$
- Because the second application requires both
  1. $R[a] = \{a, b\} \in [\![\varphi]\!]_M$
  2. $R[b] = \{b\} \in [\![\varphi]\!]_M$

Iterating

At a singleton $\{w\}$:

$$M, \{w\} \vDash \Box_{\cup}\varphi \quad \Longleftrightarrow \quad R[w] \in [\![\varphi]\!]_M$$

A second application gives:

$$M, \{w\} \vDash \Box_{\cup} \Box_{\cup} \varphi \quad \Longleftrightarrow \quad \forall v \in R[w]\ R[v] \in [\![\varphi]\!]_M$$

At each world w, we thus want:

$$R[w] \in [\![\varphi]\!]_M \quad \Longleftrightarrow \quad \forall v \in R[w]\ R[v] \in [\![\varphi]\!]_M$$

for every $[\![\varphi]\!]_M \subseteq \mathcal{P}(W)$.

If $R[w] = \varnothing$, the right-hand side is vacuously true. So idempotence forces seriality

Moreover, each successor must have exactly the same range.

The frame condition is $KD45$

The condition

$$R[w] \neq \varnothing \quad \text{and} \quad wRv \Rightarrow R[v] = R[w]$$

is equivalent to seriality, transitivity, and Euclideanness.

- Seriality: $R[w] \neq \varnothing$
- Transitivity: $wRv \Rightarrow R[v] \subseteq R[w]$
- Euclideanness: $wRv \Rightarrow R[w] \subseteq R[v]$

Hence:

$\Box_{\cup} \Box_{\cup} \varphi \equiv \Box_{\cup}\varphi$ for all φ $\quad \Longleftrightarrow \quad$ R is serial, transitive, and Euclidean.

What local modalities cannot express

- Universal-source local modalities have the shape:

$$M, T \vDash O\varphi \iff \forall w \in T\ C_\varphi(w)$$

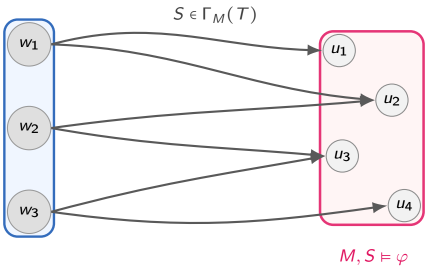
- Their strength is locality:
  modal truth at $T =$ truth at each singleton source world.
- Their limitation is the same fact: no cross-source interaction is visible.
- They cannot express coordination between successors chosen for different source worlds.
- They cannot require one joint successor team satisfying a collective property.

Global modalities

- A global modality looks at one collective successor team.
- Start with an information state $T \subseteq W$.
- The modal transition selects a single successor state $S \subseteq W$.
- Successors originating from different source worlds may interact inside S.
- For local modality, each $w \in T$ is instead checked separately.

$$M, T \vDash O\varphi \iff [\forall/\exists]S(S \in \Gamma_M(T) \;\&\; M, S \vDash \varphi),$$

where $\Gamma_M(T)$ is a set of admissible collective successor teams.



A general semantics

Let Γ_M assign to each team T a family of admissible successor teams: $\Gamma_M(T) \subseteq \mathcal{P}(W)$.

- The angelic global diamond is:

$$M, T \vDash \langle\Gamma\rangle\varphi \iff \exists S \in \Gamma_M(T)\ M, S \vDash \varphi.$$

- The robust global box is:

$$M, T \vDash [\Gamma]\varphi \iff \forall S \in \Gamma_M(T)\ M, S \vDash \varphi$$

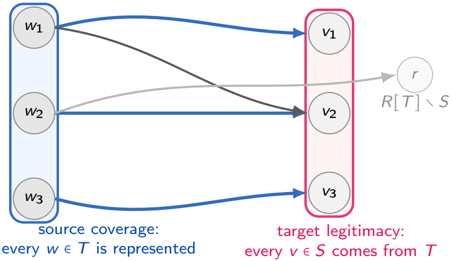
- A deterministic box can also be defined by a distinguished successor operation F_M:

$$M, T \vDash \Box_F\varphi \iff M, F_M(T) \vDash \varphi$$

Lifting accessibility to teams

Let $M = (W, R, V)$ and let $T, S \subseteq W$.

- Target legitimacy:
$$S \subseteq R[T]$$
Every successor world in S comes from some source world in T.
- Source coverage:
$$\forall w \in T\ \exists v \in S\ (wRv)$$
Every source world in T is represented in S.



Standard lax modal team semantics

- The standard lax lifting combines both:

$$TR^*S \iff S \subseteq R[T] \;\&\; \forall w \in T\, \exists v \in S\, (wRv)$$

- Equivalently, S can be obtained by choosing a nonempty $F(w) \subseteq R[w]$ for every $w \in T$ and taking $S = \bigcup_{w \in T} F(w)$.
- $M, T \vDash \Diamond\varphi \iff \exists S(TR^*S \;\&\; M, S \vDash \varphi)$
- $M, T \vDash \Box\varphi \iff M, R[T] \vDash \varphi$
- $\Diamond$ existentially selects one collective successor team.
- $\Box$ evaluates the complement at the full successor team $R[T]$.
- The two clauses are not simply existential and universal duals over the same family of successor teams.
- The strict version allows exactly one selected successor for each source world.

Dropping the source condition

- Dropping the source condition means that modality no longer requires every live possibility to have a successful continuation.
- For flat α, assuming $Y \neq \varnothing$:

$$M, X \vDash \Diamond_{\mathrm{tgt}}\alpha \Longleftrightarrow \exists w \in X\, \exists v\, (wRv \;\&\; M, v \vDash \alpha)$$

- Some live possibility has a successful continuation.
- Adding this modality to a downward-closed team logic introduces genuinely non-downward-closed state properties.

Dropping the target condition

- Every $w \in X$ must still have a successor in Y.
- But Y may also contain worlds that are not accessible from X.
- For downward-closed formulas, the target condition is actually redundant.
- For non-downward-closed φ, the equivalence can fail: unreachable worlds may provide additional witnesses.
- We therefore lose successor locality: truth may depend on worlds outside the part of the model reachable from X.

Global modality $\Box$ and idempotence

$$M, T \vDash \Box\varphi \iff M, R[T] \vDash \varphi$$

It first forms one collective successor team $R[T] = \bigcup_{w \in T} R[w]$ and then evaluates the complement once.

Applying it twice gives:

$$M, T \vDash \Box\Box\varphi \iff M, R^2[T] \vDash \varphi$$

So idempotence for all state properties requires exactly, as in standard modal logic:

$$R^2[T] = R[T] \text{ for every } T \subseteq W$$

Robust box versus full box

Assume every $w \in T$ has at least one successor, so $R[T]$ is itself admissible.

- Robust box entails full box: $[\Gamma_{\mathrm{lax}}]\varphi \vDash \Box\varphi$
- Full box entails robust box when φ is downward closed: $\Box\varphi \vDash [\Gamma_{\mathrm{lax}}]\varphi$
- For non-downward-closed complements, the full team may have a property that some smaller admissible successor team lacks.

What global modalities can express

Because the complement is evaluated on one joint successor team, global modalities can express collective successor properties.

- The successor team contains both p- and $\neg p$-possibilities.
- A dependency holds across the selected successor possibilities: $\Diamond \mathrm{dep}(p, q)$
- These are properties of a set of successor possibilities, not merely properties of each successor world separately.

What global modalities forget

Standard global modality remembers the output team S, but not how it was generated.

- A lax witness can be represented by: $w \mapsto F(w) \subseteq R[w]$
- Standard semantics keeps only: $S = \bigcup_{w \in T} F(w)$
- It forgets which source world produced which successor world.
- Therefore it cannot directly express properties such as:
  - every source world has two distinguishable continuations
  - two source worlds converge on the same successor
  - target facts depend on source facts
- To express those, one needs something like path-based refinements.

Bonus: modals as quantifiers

- Recall that in Lecture 4, we examined a treatment of modality as existential quantifiers over a world variable.
- So let assignments carry a current-world variable v. Accessibility can be used to bind a fresh successor variable w.
- The relativized universal modal quantifier is:
  $X[R(v)/w] = \{s[a/w] : s \in X \text{ and } s(v)Ra\}$.
- $M, X \vDash \Box_{v \to w} \psi \quad \Longleftrightarrow \quad M, X[R(v)/w] \vDash \psi$
- The lax existential modal quantifier is:
  $F : X \to \mathcal{P}^+(W), \qquad \varnothing \neq F(s) \subseteq R[s(v)]$.
- $M, X \vDash \Diamond_{v \to w} \psi \quad \Longleftrightarrow \quad \exists F\; M, X[F/w] \vDash \psi$.

Relation to global modalities

- Let $T = X(v)$ and let $\pi_w(Y) = Y(w)$ be the set of w-values in a team Y.
- For a lax witness F, the induced successor state is: $S_F = \pi_w(X[F/w]) = \bigcup_{s \in X} F(s)$.
- The possible sets S_F are exactly the standard lax successors: $TR^*S \quad \Longleftrightarrow \quad S \subseteq R[T] \;\&\; \forall a \in T\, \exists b \in S\, (aRb)$
- Hence, if ψ depends only on the w-projection, then $\Diamond_{v \to w}\psi$ matches $\langle \Gamma_\ell \rangle \varphi$.
- Similarly, $\Box_{v \to w}\psi$ matches $\Box\,\varphi$ whenever ψ depends only on $\pi_w(X[R(v)/w]) = R[T]$.

Invariance

- The collapse to global modality requires the matrix to forget the source.
- Roughly, for all teams Y, Z over v, w,
  $Y(w) = Z(w) \Longrightarrow (M, Y \vDash \psi \iff M, Z \vDash \psi)$
- First-order formulas in w alone satisfy this condition.
- Dependence and variation atoms mentioning both v and w usually do not.
- Quantifier modals preserve provenance: which source assignment generated which successor.
- Global modalities keep only the collective successor state S.

Iteration

- Use a fresh world variable at every modal step:
  $v_0 := v, \qquad v_1 := w, \qquad v_2 := u, \; \ldots.$
- The relativized modal quantifiers are applied successively.
- Thus two universal steps generate a team of paths:
  $X[R(v_0)/v_1][R(v_1)/v_2] = \{s[a/v_1][b/v_2] : s(v_0)Ra \text{ and } aRb\}$
- The endpoint is v_2, but the team still records the intermediate world v_1 and the initial world v_0.
- Projecting to the newest variable v_i recovers endpoint-based global modal semantics. Retaining all variables yields a stronger path-sensitive semantics.

Modality need not be metaphysical

- A modal operator in team semantics can quantify over information states
- In Kripke-style modality, the relevant transition is usually external: wRv.
- In information-change modality, the transition may be internal to the current state: $S \subseteq T$ or $S \supseteq T$
- We have already seen this with the black diamond modality.
- Refinement corresponds to gaining information: fewer worlds remain live.
- Expansion corresponds to losing information: more worlds become live.
- We can model epistemic-like dynamic modalities.

Four order modalities

- Refinement possibility:

$$M, T \vDash \langle\downarrow\rangle\varphi \iff \exists S \subseteq T\,(S \neq \varnothing \;\&\; M, S \vDash \varphi)$$

- Robust refinement:

$$M, T \vDash [\downarrow]\varphi \iff \forall S \subseteq T\; M, S \vDash \varphi$$

- Information-loss possibility:

$$M, T \vDash \langle\uparrow\rangle\varphi \iff \exists S \supseteq T\; M, S \vDash \varphi$$

- Robust information loss:

$$M, T \vDash [\uparrow]\varphi \iff \forall S \supseteq T\; M, S \vDash \varphi$$

Algebra of state properties

$$P_\varphi = \{T \subseteq W : M, T \vDash \varphi\}$$

The order modalities transform P_φ as follows:

- Refinement possibility gives the nonempty upward closure:

$$\langle\downarrow\rangle P = \{T : \exists S \neq \varnothing\,(S \subseteq T \ \&\ S \in P)\}$$

- Robust refinement gives the downward interior:

$$[\downarrow]P = \{T : \forall S \subseteq T\ S \in P\}$$

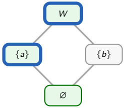
- Information-loss possibility gives the downward closure:

$$\langle\uparrow\rangle P = \{T : \exists S \supseteq T\ S \in P\}$$

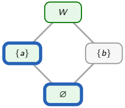
- Robust information loss gives the upward interior:

$$[\uparrow]P = \{T : \forall S \supseteq T\ S \in P\}$$

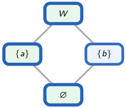
nonempty upward closure



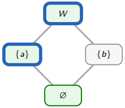
downward interior



downward closure



upward interior



An interesting fact

$$X = \mathcal{P}(W) \qquad X^{+} = X \setminus \{\varnothing\} \qquad P \subseteq X$$

- The four order modalities act on state properties $P \subseteq X$:

$$\begin{array}{lcllcl} \mathsf{E}(P) & = & \uparrow (P \cap X^{+}) & \mathsf{L}(P) & = & \downarrow P \\ \mathsf{R}(P) & = & \{T : \downarrow T \subseteq P\} & \mathsf{U}(P) & = & \{T : \uparrow T \subseteq P\}. \end{array}$$

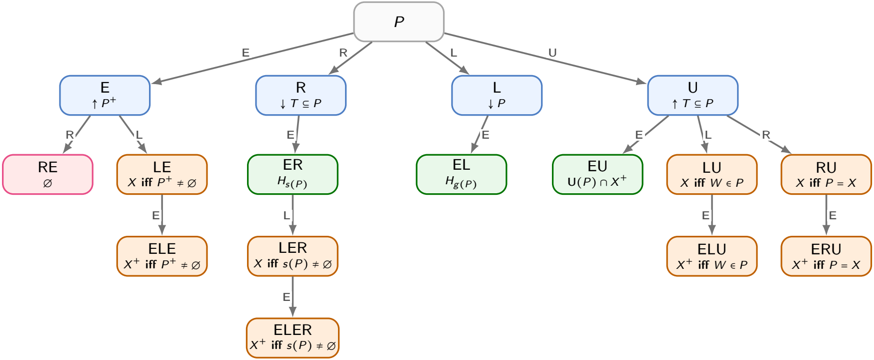
- A nesting such as $[\uparrow]\langle\downarrow\rangle[\downarrow]\langle\uparrow\rangle\varphi$ corresponds to UERL(P_{φ}).
- Although there are infinitely many syntactic nestings, many define the same transformation $\mathcal{P}(X) \to \mathcal{P}(X)$.

An interesting fact

- If $|W| \geq 2$, the nonempty composites generated by $\mathrm{E}, \mathrm{R}, \mathrm{L}, \mathrm{U}$ give exactly sixteen maps.
- The collapse is driven by a few invariants:
  - $g(P) = \bigcup P$: the ground team: the worlds occurring somewhere in P.
  - $s(P) = \{w : \varnothing \in P \text{ and } \{w\} \in P\}$: the safe singletons: worlds w such that both $\varnothing$ and $\{w\}$ are in P.
  - $H_A = \{T : T \cap A \neq \varnothing\}$: the hitting property generated by a set of worlds A.
- Any fixed nesting pattern stabilizes after at most two repetitions.

The sixteen maps

$$P^+ := P \cap X^+ \qquad H_A := \{T : T \cap A \neq \varnothing\} \qquad s(P) := \{w : \varnothing \in P, \{w\} \in P\}$$



Exercise: Bicompleteness

Consider the propositional team logic with the following syntax:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbin{\vee\!\!\!\vee} \phi \mid \mathrm{ne}^*$$

And the following semantics:

$$
\begin{array}{lll}
s \vDash p & :\Longleftrightarrow & \text{for all } w \in s : w(p) = 1 \\
s \dashv p & :\Longleftrightarrow & \text{for all } w \in s : w(p) = 0 \\
s \vDash \bot & :\Longleftrightarrow & s = \varnothing \\
s \dashv \bot & & \text{always the case} \\
s \vDash \mathrm{ne}^* & :\Longleftrightarrow & s \neq \varnothing \\
s \dashv \mathrm{ne}^* & :\Longleftrightarrow & s \neq \varnothing \\
s \vDash \neg\phi & :\Longleftrightarrow & s \dashv \phi \\
s \dashv \neg\phi & :\Longleftrightarrow & s \vDash \phi \\
s \vDash \phi \wedge \psi & :\Longleftrightarrow & s \vDash \phi \text{ and } s \vDash \psi \\
s \dashv \phi \wedge \psi & :\Longleftrightarrow & \exists t, u \text{ s.t. } s = t \cup u \text{ and } t \dashv \phi \text{ and } u \dashv \psi \\
s \vDash \phi \vee \psi & :\Longleftrightarrow & \exists t, u \text{ s.t. } s = t \cup u \text{ and } t \vDash \phi \text{ and } u \vDash \psi \\
s \dashv \phi \vee \psi & :\Longleftrightarrow & s \dashv \phi \text{ and } s \dashv \psi \\
s \vDash \phi \mathbin{\vee\!\!\!\vee} \psi & :\Longleftrightarrow & s \vDash \phi \text{ or } s \vDash \psi \\
s \dashv \phi \mathbin{\vee\!\!\!\vee} \psi & :\Longleftrightarrow & s \dashv \phi \text{ and } s \dashv \psi
\end{array}
$$

(Continued on the next slide.)

Exercise: Bicompleteness

Denote $\phi \equiv^{\pm} \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$.

Show that:

1. There is a formula ϕ_{ne} with $\phi_{\mathsf{ne}} \equiv^{\pm} \mathsf{ne}$ (i.e., $t \vDash \phi_{\mathsf{ne}} \iff t \neq \varnothing$ and $t \dashv \phi_{\mathsf{ne}} \iff t = \varnothing$).
2. There is a formula $\phi_{\perp\!\!\!\perp}$ s.t. $t \vDash \phi_{\perp\!\!\!\perp}$ never holds, and $\neg\phi_{\perp\!\!\!\perp} \equiv \top$.
3. For each ϕ, there is a ϕ^* s.t. $\phi \equiv \phi^*$, and $\neg\phi^*$ has the empty team property (hint: make use of the formula ϕ_{ne} from 1).
4. For each ϕ, there is a $\phi^{\top}$ s.t. $\phi^{\top} \equiv \phi$, and $\neg\phi^{\top} \equiv \top$ (hint: make use of the formula ϕ^* from 3).
5. This logic is bicomplete for all pairs. I.e., for all ϕ, ψ, there is a θ s.t. $\phi \equiv \theta$ and $\psi \equiv \neg\theta$. (hint: make use of the formula $\phi_{\perp\!\!\!\perp}$ from 2, and the formulas $\phi_{\top}$ from 4).